Peeter Joot

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1 Angle between geometric elements.

Have the calculation for the angle between bivectors done elsewhere

$$\cos\theta = -\frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|} \tag{1}$$

For $\theta \in [0, \pi]$.

The vector/vector result is well known and also works fine in $\mathbb{R}^N$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} \tag{2}$$

2 Calculation for a line and a plane.

Given a line with unit direction vector $\mathbf{u}$, and plane with unit direction bivector $\mathbf{A}$, the component of that vector in the plane is:

$$-\mathbf{u} \cdot \mathbf{A}\mathbf{A}.$$

So the direction cosine is available immediately

$$\cos\theta = \mathbf{u} \cdot \frac{-\mathbf{u} \cdot \mathbf{A}\mathbf{A}}{|\mathbf{u} \cdot \mathbf{A}\mathbf{A}|}$$

However, this can be reduced significantly. Start with the denominator

$$\begin{aligned}
|\mathbf{u} \cdot \mathbf{A}\mathbf{A}|^2 &= (\mathbf{u} \cdot \mathbf{A}\mathbf{A})(\mathbf{A}\mathbf{A} \cdot \mathbf{u}) \\
&= (\mathbf{u} \cdot \mathbf{A})^2.
\end{aligned}$$

And in the numerator we have:

$$
\begin{aligned}
\mathbf{u} \cdot (\mathbf{u} \cdot \mathbf{A}\mathbf{A}) &= \frac{1}{2}(\mathbf{u}(\mathbf{u} \cdot \mathbf{A}\mathbf{A}) + (\mathbf{u} \cdot \mathbf{A}\mathbf{A})\mathbf{u}) \\
&= \frac{1}{2}((\mathbf{u}\mathbf{u} \cdot \mathbf{A})\mathbf{A} + (\mathbf{u} \cdot \mathbf{A})\mathbf{A}\mathbf{u}) \\
&= \frac{1}{2}((\mathbf{A} \cdot \mathbf{u}\mathbf{u})\mathbf{A} - (\mathbf{A} \cdot \mathbf{u})\mathbf{A}\mathbf{u}) \\
&= (\mathbf{A} \cdot \mathbf{u})\frac{1}{2}(\mathbf{u}\mathbf{A} - \mathbf{A}\mathbf{u}) \\
&= -(\mathbf{A} \cdot \mathbf{u})^2.
\end{aligned}
$$

Putting things back together

$$
\cos\theta = \frac{(\mathbf{A} \cdot \mathbf{u})^2}{|\mathbf{u} \cdot \mathbf{A}|} = |\mathbf{u} \cdot \mathbf{A}|
$$

The strictly positive value here is consistent with the fact that theta as calculated is in the $[0, \pi/2]$ range.

Restated for consistency with equations 2 and 1 in terms of not neccessarily unit vector and bivectors **u** and **A**, we have

$$
\cos\theta = \frac{|\mathbf{u} \cdot \mathbf{A}|}{|\mathbf{u}||\mathbf{A}|} \tag{3}
$$