Radial decomposition of angular velocity and angular velocity.

Peeter Joot

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A derivation of the radial velocity decomposition for a physics forum poster who couldn't understand my GA approach or use it as a hint for his own derivation.

1.1

Starting point is taking derivatives of:

$$\mathbf{r} = r\hat{\mathbf{r}}$$

$$\mathbf{v} = r'\hat{\mathbf{r}} + r\hat{\mathbf{r}}'$$

It can be shown (see for example, Salus and Hille, "Calculus") that the unit vector derivative can be expressed using the cross product:

$$\hat{\mathbf{r}}' = \frac{1}{r}\left(\hat{\mathbf{r}} \times \frac{d\mathbf{r}}{dt}\right) \times \hat{\mathbf{r}}.$$

Now, one can express r' in terms of $\mathbf{r}$ as well as follows:

$$(\mathbf{r} \cdot \mathbf{r})' = 2\mathbf{v} \cdot \mathbf{r} = 2rr'.$$

Thus the derivative of the vector magnitude is part of a projective term:

$$r' = \hat{\mathbf{r}} \cdot \mathbf{v}.$$

Putting this together one has velocity in terms of projective and rejective components along a radial direction:

$$\mathbf{v} = (\hat{\mathbf{r}} \cdot \mathbf{v})\,\hat{\mathbf{r}} + \left(\hat{\mathbf{r}} \times \frac{d\mathbf{r}}{dt}\right) \times \hat{\mathbf{r}}.$$

Now $\omega = \frac{\mathbf{r}\times\mathbf{v}}{r^2}$ term is what we call the angular velocity. The magnitude of this is the rate of change of the angle between the radial arm and the direction of rotation. The direction of this cross product is normal to the plane of rotation and encodes both the rotational plane and the direction of the rotation. Putting these together one has the total velocity expressed radially:

$$\mathbf{v} = (\hat{\mathbf{r}} \cdot \mathbf{v})\,\hat{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}.$$

1.2 Acceleration.

Acceleration follows in the same fashion.

$$\begin{aligned}
\mathbf{v}' &= (\underbrace{(\hat{\mathbf{r}} \cdot \mathbf{v})\,\hat{\mathbf{r}}}_{r'\hat{\mathbf{r}}})' + (\underbrace{\boldsymbol{\omega} \times \mathbf{r}}_{(\mathbf{r}\times\mathbf{v})\times\frac{\mathbf{r}}{r^2}})' \\
&= r''\hat{\mathbf{r}} + r'\frac{\boldsymbol{\omega} \times \mathbf{r}}{r} + (\hat{\mathbf{r}} \times \mathbf{a}) \times \hat{\mathbf{r}} + \underbrace{(\mathbf{v} \times \mathbf{v})}_{=0} \times \frac{\mathbf{r}}{r^2} + (\mathbf{r} \times \mathbf{v}) \times \left(\frac{\mathbf{r}}{r^2}\right)'
\end{aligned}$$

That last derivative is

$$\begin{aligned}
\left(\frac{\mathbf{r}}{r^2}\right)' &= \left(\frac{\hat{\mathbf{r}}}{r}\right)' \\
&= \frac{\hat{\mathbf{r}}'}{r} - \frac{\hat{\mathbf{r}}r'}{r^2} \\
&= \frac{\boldsymbol{\omega} \times \mathbf{r}}{r^2} - \frac{\hat{\mathbf{r}}r'}{r^2},
\end{aligned}$$

and back substuition gives:

$$\mathbf{v}' = r''\hat{\mathbf{r}} + r'\frac{\boldsymbol{\omega} \times \mathbf{r}}{r} + (\hat{\mathbf{r}} \times \mathbf{a}) \times \hat{\mathbf{r}} + (\mathbf{r} \times \mathbf{v}) \times \left(\frac{\boldsymbol{\omega} \times \mathbf{r}}{r^2} - \frac{\hat{\mathbf{r}}r'}{r^2}\right).$$

Cancelling terms and collecting we have the final result for acceleration expressed radially:

$$\mathbf{v}' = \mathbf{a} = r''\hat{\mathbf{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + (\hat{\mathbf{r}} \times \mathbf{a}) \times \hat{\mathbf{r}} \tag{1}$$

Now, applying the angular velocity via cross product takes the vector back to the original plane, but inverts it. Thus we can write the acceleration completely in terms of the radially directed components, and the perpendicular component.

$$\mathbf{a} = r''\hat{\mathbf{r}} - \mathbf{r}\omega^2 + (\hat{\mathbf{r}} \times \mathbf{a}) \times \hat{\mathbf{r}} \tag{2}$$

An alternate way to express this is in terms of radial scalar acceleration:

$$\mathbf{a} \cdot \hat{\mathbf{r}} = r'' - r\omega^2. \tag{3}$$

This is the acceleration analogue of the scalar radial velocity component demonstrated above:

$$\mathbf{v} \cdot \hat{\mathbf{r}} = r'. \tag{4}$$