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1 General triple product reduction formula.

Consideration of the reciprocal frame bivector decomposition required the following identity

$$(\mathbf{A}_a \wedge \mathbf{A}_b) \cdot \mathbf{A}_c = \mathbf{A}_a \cdot (\mathbf{A}_b \cdot \mathbf{A}_c) \tag{1}$$

This holds when $a + b \le c$, and $a <= b$. Similar equations for vector wedge blade dot blade reduction can be found in NFCM, but intuition let me to believe the above generalization was valid.

To prove this use the definition of the generalized dot product of two blades:

$$(\mathbf{A}_a \wedge \mathbf{A}_b) \cdot \mathbf{A}_c = \langle (\mathbf{A}_a \wedge \mathbf{A}_b)\mathbf{A}_c \rangle_{|c-(a+b)|}$$

The subsequent discussion is restricted to the $b \ge a$ case. Would have to think whether this restriction is required.

$$\begin{aligned}
\mathbf{A}_a \wedge \mathbf{A}_b &= \mathbf{A}_a\mathbf{A}_b - \sum_{i=|b-a|, i+=2}^{a+b} \langle \mathbf{A}_a\mathbf{A}_b \rangle_i \\
&= \mathbf{A}_a\mathbf{A}_b - \sum_{k=0}^{a-1} \langle \mathbf{A}_a\mathbf{A}_b \rangle_{2k+b-a}
\end{aligned}$$

Back substitution gives:

$$\langle (\mathbf{A}_a \wedge \mathbf{A}_b)\mathbf{A}_c \rangle_{|c-(a+b)|} = \langle \mathbf{A}_a\mathbf{A}_b\mathbf{A}_c \rangle_{|c-(a+b)|} - \sum_{k=0}^{a-1} \left\langle \langle \mathbf{A}_a\mathbf{A}_b \rangle_{2k+b-a} \mathbf{A}_c \right\rangle_{c-a-b}$$

Temporarily writing $\langle \mathbf{A}_a \mathbf{A}_b \rangle_{2k+b-a} = \mathbf{C}_i$,

$$\begin{aligned}
\langle \mathbf{A}_a \mathbf{A}_b \rangle_{2k+b-a} \mathbf{A}_c &= \sum_{j=c-i, j+=2}^{c+i} \langle \mathbf{C}_i \mathbf{A}_c \rangle_j \\
&= \sum_{r=0}^{i} \langle \mathbf{C}_i \mathbf{A}_c \rangle_{c-i+2r} \\
&= \sum_{r=0}^{2k+b-a} \langle \mathbf{C}_i \mathbf{A}_c \rangle_{c-2k-b+a+2r} \\
&= \sum_{r=0}^{2k+b-a} \langle \mathbf{C}_i \mathbf{A}_c \rangle_{c-b+a+2(r-k)}
\end{aligned}$$

We want the only the following grade terms:

$$c - b + a + 2(r-k) = c - b - a \implies r = k - a$$

There are many such k, r combinations, but we have a $k \in [0, a-1]$ constraint, which implies $r \in [-a, -1]$. This contradicts with r strictly positive, so there are no such grade elements.

This gives an intermediate result, the reduction of the triple product to a direct product, removing the explicit wedge:

$$(\mathbf{A}_a \wedge \mathbf{A}_b) \cdot \mathbf{A}_c = \langle \mathbf{A}_a \mathbf{A}_b \mathbf{A}_c \rangle_{c-a-b} \tag{2}$$

$$\begin{aligned}
\langle \mathbf{A}_a \mathbf{A}_b \mathbf{A}_c \rangle_{c-a-b} &= \langle \mathbf{A}_a (\mathbf{A}_b \mathbf{A}_c) \rangle_{c-a-b} \\
&= \left\langle \mathbf{A}_a \sum_i \langle \mathbf{A}_b \mathbf{A}_c \rangle_i \right\rangle_{c-a-b} \\
&= \left\langle \sum_j \left\langle \mathbf{A}_a \sum_i \langle \mathbf{A}_b \mathbf{A}_c \rangle_i \right\rangle_j \right\rangle_{c-a-b}
\end{aligned}$$

Explicitly specifying the grades here is omitted for simplicity. The lowest grade of these is $(c-b) - a$, and all others are higher, so grade selection excludes them.

By definition

$$\langle \mathbf{A}_b \mathbf{A}_c \rangle_{c-b} = \mathbf{A}_b \cdot \mathbf{A}_c$$

so that lowest grade term is thus

$$\left\langle \mathbf{A}_a \langle \mathbf{A}_b \mathbf{A}_c \rangle_{c-b} \right\rangle_{c-a-b} = \langle \mathbf{A}_a (\mathbf{A}_b \cdot \mathbf{A}_c) \rangle_{c-a-b} = \mathbf{A}_a \cdot (\mathbf{A}_b \cdot \mathbf{A}_c)$$

This completes the proof.

2 reduction of grade of dot product of two blades.

The result above can be applied to reducing the dot product of two blades. For $k <= s$:

$$
\begin{aligned}
&(\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3 \cdots \wedge \mathbf{a}_k) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \cdots \wedge \mathbf{b}_s) \\
&= (\mathbf{a}_1 \wedge (\mathbf{a}_2 \wedge \mathbf{a}_3 \cdots \wedge \mathbf{a}_k)) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \cdots \wedge \mathbf{b}_s) \\
&= (\mathbf{a}_1 \cdot ((\mathbf{a}_2 \wedge \mathbf{a}_3 \cdots \wedge \mathbf{a}_k)) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \cdots \wedge \mathbf{b}_s)) \\
&= (\mathbf{a}_1 \cdot (\mathbf{a}_2 \cdot (\mathbf{a}_3 \cdots \wedge \mathbf{a}_k)) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \cdots \wedge \mathbf{b}_s)) \\
&= \cdots \\
&= \mathbf{a}_1 \cdot (\mathbf{a}_2 \cdot (\mathbf{a}_3 \cdot (\cdots \cdot (\mathbf{a}_k \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \cdots \wedge \mathbf{b}_s)))))
\end{aligned}
$$

This can be reduced to a single determinant, as is done in the Flanders' differential forms book definition of the $\bigwedge^k$ inner product (which is then used to define the hodge dual).

The first such product is:

$$\mathbf{a}_k \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \cdots \wedge \mathbf{b}_k) = \sum (-1)^{u-1} (\mathbf{a}_k \cdot \mathbf{b}_u) \mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_u \cdots \wedge \mathbf{b}_k$$

Next, take dot product with $\mathbf{a}_{k-1}$:

1. $k = 2$

$$
\begin{aligned}
\mathbf{a}_{k-1} \cdot (\mathbf{a}_k \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \cdots \wedge \mathbf{b}_k)) &= \sum_{v \neq u} (-1)^{u-1} (\mathbf{a}_k \cdot \mathbf{b}_u)(\mathbf{a}_1 \cdot \mathbf{b}_v) \\
&= \sum_{u<v} (-1)^{v-1} (\mathbf{a}_k \cdot \mathbf{b}_v)(\mathbf{a}_1 \cdot \mathbf{b}_u) + \sum_{u<v} (-1)^{u-1} (\mathbf{a}_k \cdot \mathbf{b}_u)(\mathbf{a}_1 \cdot \mathbf{b}_v) \\
&= + \sum_{u<v} (\mathbf{a}_k \cdot \mathbf{b}_u)(\mathbf{a}_1 \cdot \mathbf{b}_v) - \sum_{u<v} (\mathbf{a}_k \cdot \mathbf{b}_v)(\mathbf{a}_1 \cdot \mathbf{b}_u) \\
&= + \sum_{u<v} (\mathbf{a}_k \cdot \mathbf{b}_u)(\mathbf{a}_1 \cdot \mathbf{b}_v) - (\mathbf{a}_k \cdot \mathbf{b}_v)(\mathbf{a}_1 \cdot \mathbf{b}_u)
\end{aligned}
$$

$$- \sum_{u<v} \begin{vmatrix} \mathbf{a}_{k-1} \cdot \mathbf{b}_u & \mathbf{a}_{k-1} \cdot \mathbf{b}_v \\ \mathbf{a}_k \cdot \mathbf{b}_u & \mathbf{a}_k \cdot \mathbf{b}_v \end{vmatrix} \tag{3}$$

2. $k > 2$

$$\mathbf{a}_{k-1} \cdot (\mathbf{a}_k \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \cdots \wedge \mathbf{b}_k))$$

$$
\begin{aligned}
&= \sum (-1)^{u-1}(\mathbf{a}_k \cdot \mathbf{b}_u)\mathbf{a}_{k-1} \cdot (\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_u \cdots \wedge \mathbf{b}_k) \\
&= \sum_{v<u} (-1)^{u-1}(\mathbf{a}_k \cdot \mathbf{b}_u)(-1)^{v-1}(\mathbf{a}_{k-1} \cdot \mathbf{b}_v)(\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_v \cdots \check{\mathbf{b}}_u \cdots \wedge \mathbf{b}_k) \\
&+ \sum_{v>u} (-1)^{u-1}(\mathbf{a}_k \cdot \mathbf{b}_u)(-1)^{v}(\mathbf{a}_{k-1} \cdot \mathbf{b}_v)(\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_u \cdots \check{\mathbf{b}}_v \cdots \wedge \mathbf{b}_k)
\end{aligned}
$$

Add negation exponents, and use a change of variables for the first sum

$$
\begin{aligned}
&= \sum_{u<v} (-1)^{v+u}(\mathbf{a}_k \cdot \mathbf{b}_v)(\mathbf{a}_{k-1} \cdot \mathbf{b}_u)(\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_u \cdots \check{\mathbf{b}}_v \cdots \wedge \mathbf{b}_k) \\
&- \sum_{u<v} (-1)^{u+v}(\mathbf{a}_k \cdot \mathbf{b}_u)(\mathbf{a}_{k-1} \cdot \mathbf{b}_v)(\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_u \cdots \check{\mathbf{b}}_v \cdots \wedge \mathbf{b}_k)
\end{aligned}
$$

Merge sums:

$$
\begin{aligned}
&= \sum_{u<v} (-1)^{u+v} \left((\mathbf{a}_k \cdot \mathbf{b}_v)(\mathbf{a}_{k-1} \cdot \mathbf{b}_u) - (\mathbf{a}_k \cdot \mathbf{b}_u)(\mathbf{a}_{k-1} \cdot \mathbf{b}_v)\right) \\
&(\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_u \cdots \check{\mathbf{b}}_v \cdots \wedge \mathbf{b}_k)
\end{aligned}
$$

$$
\begin{gathered}
\mathbf{a}_{k-1} \cdot (\mathbf{a}_k \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \cdots \wedge \mathbf{b}_k)) = \\
\sum_{u<v} (-1)^{u+v} \begin{vmatrix} \mathbf{a}_{k-1} \cdot \mathbf{b}_u & \mathbf{a}_{k-1} \cdot \mathbf{b}_v \\ \mathbf{a}_k \cdot \mathbf{b}_u & \mathbf{a}_k \cdot \mathbf{b}_v \end{vmatrix} (\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_u \cdots \check{\mathbf{b}}_v \cdots \wedge \mathbf{b}_k)
\end{gathered}
\tag{4}
$$

Note that special casing $k = 2$ doesn't seem to be required because in that case $-1^{u+v} = -1^{1+2} = -1$, so this is identical to equation 3 afterall.

2.1 Pause to reflect.

Although my initial aim was to show that $\mathbf{A}_k \cdot \mathbf{B}_k$ could be expressed as a determinant as in the differential forms book (different sign though), and to determine exactly what that determinant is, there are some useful identities that fall out of this even just for this bivector kvector dot product expansion.

Here's a summary of some of the things figured out so far

1. Dot product of grade one blades.

   Here we have a result that can be expressed as a one by one determinant. Worth mentioning to explicitly show the sign.

$$
\mathbf{a} \cdot \mathbf{b} = \det[\mathbf{a} \cdot \mathbf{b}] \tag{5}
$$

2. Dot product of grade two blades.

$$
(\mathbf{a}_1 \wedge \mathbf{a}_2) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2) = - \begin{vmatrix} \mathbf{a}_1 \cdot \mathbf{b}_1 & \mathbf{a}_1 \cdot \mathbf{b}_2 \\ \mathbf{a}_2 \cdot \mathbf{b}_1 & \mathbf{a}_2 \cdot \mathbf{b}_2 \end{vmatrix} = -\det[\mathbf{a}_i \cdot \mathbf{b}_j] \tag{6}
$$

3. Dot product of grade two blade with grade > 2 blade.

$$
\begin{aligned}
&(\mathbf{a}_1 \wedge \mathbf{a}_2) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \cdots \wedge \mathbf{b}_k) \\
&= \sum_{u<v} (-1)^{u+v-1} (\mathbf{a}_1 \wedge \mathbf{a}_2) \cdot (\mathbf{b}_u \wedge \mathbf{b}_v)(\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_u \cdots \check{\mathbf{b}}_v \cdots \wedge \mathbf{b}_k)
\end{aligned}
\tag{7}
$$

Observe how similar this is to the vector blade dot product expansion:

$$
\mathbf{a} \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \cdots \wedge \mathbf{b}_k) = \sum (-1)^{i-1} (\mathbf{a} \cdot \mathbf{b}_i)(\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_i \cdots \wedge \mathbf{b}_k) \tag{8}
$$

2.1.1 Expand it for $k = 3$

Explicit expansion of equation 7 for the $k = 3$ case, is also helpful to get a feel for the equation:

$$
\begin{aligned}
(\mathbf{a}_1 \wedge \mathbf{a}_2) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \mathbf{b}_3) &= (\mathbf{a}_1 \wedge \mathbf{a}_2) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2)\mathbf{b}_3 \\
&+ (\mathbf{a}_1 \wedge \mathbf{a}_2) \cdot (\mathbf{b}_3 \wedge \mathbf{b}_1)\mathbf{b}_2 \\
&+ (\mathbf{a}_1 \wedge \mathbf{a}_2) \cdot (\mathbf{b}_2 \wedge \mathbf{b}_3)\mathbf{b}_1
\end{aligned}
$$

Observe the cross product like alternation in sign and indexes. This suggests that a more natural way to express the sign coefficient may be via a $\mathrm{sgn}(\pi)$ expression for the sign of the permutation of indexes.

3 trivector dot product.

With the result of equation 7, or the earlier equivalent determinant expression in equation 4 we are now in a position to evaluate the dot product of a trivector and a greater or equal grade blade.

$$
\mathbf{a}_1 \cdot ((\mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \cdots \wedge \mathbf{b}_k))
$$

$$
\begin{aligned}
&= \sum_{u<v} (-1)^{u+v-1} (\mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_u \wedge \mathbf{b}_v)\mathbf{a}_1 \cdot (\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_u \cdots \check{\mathbf{b}}_v \cdots \wedge \mathbf{b}_k) \\
&= \sum_{w<u<v} (-1)^{u+v+w} (\mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_u \wedge \mathbf{b}_v)(\mathbf{a}_1 \cdot \mathbf{b}_w)(\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_w \cdots \check{\mathbf{b}}_u \cdots \check{\mathbf{b}}_v \cdots \wedge \mathbf{b}_k) \\
&+ \sum_{u<w<v} (-1)^{u+v+w-1} (\mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_u \wedge \mathbf{b}_v)(\mathbf{a}_1 \cdot \mathbf{b}_w)(\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_u \cdots \check{\mathbf{b}}_w \cdots \check{\mathbf{b}}_v \cdots \wedge \mathbf{b}_k) \\
&+ \sum_{u<v<w} (-1)^{u+v+w} (\mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_u \wedge \mathbf{b}_v)(\mathbf{a}_1 \cdot \mathbf{b}_w)(\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_u \cdots \check{\mathbf{b}}_v \cdots \check{\mathbf{b}}_w \cdots \wedge \mathbf{b}_k)
\end{aligned}
$$

Change the indexes of summation and grouping like terms we have:

$$\sum_{u<v<w} (-1)^{u+v+w} \Big((\mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_v \wedge \mathbf{b}_w)(\mathbf{a}_1 \cdot \mathbf{b}_u) \\ - (\mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_u \wedge \mathbf{b}_w)(\mathbf{a}_1 \cdot \mathbf{b}_v) \\ + (\mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_u \wedge \mathbf{b}_v)(\mathbf{a}_1 \cdot \mathbf{b}_w) \\ \Big)(\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_u \cdots \check{\mathbf{b}}_v \cdots \check{\mathbf{b}}_w \cdots \wedge \mathbf{b}_k)$$

Now, each of the embbedded dot products were in fact determinants:

$$(\mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_x \wedge \mathbf{b}_y) = - \begin{vmatrix} \mathbf{a}_2 \cdot \mathbf{b}_x & \mathbf{a}_2 \cdot \mathbf{b}_y \\ \mathbf{a}_3 \cdot \mathbf{b}_x & \mathbf{a}_3 \cdot \mathbf{b}_y \end{vmatrix}$$

Thus, we can expand these triple dot products like so (factor of -1 omitted):

$$\begin{aligned}
&(\mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_v \wedge \mathbf{b}_w)(\mathbf{a}_1 \cdot \mathbf{b}_u) \\
&- (\mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_u \wedge \mathbf{b}_w)(\mathbf{a}_1 \cdot \mathbf{b}_v) \\
&+ (\mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_u \wedge \mathbf{b}_v)(\mathbf{a}_1 \cdot \mathbf{b}_w) \\
&= (\mathbf{a}_1 \cdot \mathbf{b}_u) \begin{vmatrix} \mathbf{a}_2 \cdot \mathbf{b}_v & \mathbf{a}_2 \cdot \mathbf{b}_w \\ \mathbf{a}_3 \cdot \mathbf{b}_v & \mathbf{a}_3 \cdot \mathbf{b}_w \end{vmatrix} \\
&- (\mathbf{a}_1 \cdot \mathbf{b}_v) \begin{vmatrix} \mathbf{a}_2 \cdot \mathbf{b}_u & \mathbf{a}_2 \cdot \mathbf{b}_w \\ \mathbf{a}_3 \cdot \mathbf{b}_u & \mathbf{a}_3 \cdot \mathbf{b}_w \end{vmatrix} \\
&+ (\mathbf{a}_1 \cdot \mathbf{b}_w) \begin{vmatrix} \mathbf{a}_2 \cdot \mathbf{b}_u & \mathbf{a}_2 \cdot \mathbf{b}_v \\ \mathbf{a}_3 \cdot \mathbf{b}_u & \mathbf{a}_3 \cdot \mathbf{b}_v \end{vmatrix} \\
&= \begin{vmatrix} \mathbf{a}_1 \cdot \mathbf{b}_u & \mathbf{a}_1 \cdot \mathbf{b}_v & \mathbf{a}_1 \cdot \mathbf{b}_w \\ \mathbf{a}_2 \cdot \mathbf{b}_u & \mathbf{a}_2 \cdot \mathbf{b}_v & \mathbf{a}_2 \cdot \mathbf{b}_w \\ \mathbf{a}_3 \cdot \mathbf{b}_u & \mathbf{a}_3 \cdot \mathbf{b}_v & \mathbf{a}_3 \cdot \mathbf{b}_w \end{vmatrix}
\end{aligned}$$

Final back substition gives:

$$\begin{aligned}
&(\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \cdots \wedge \mathbf{b}_k) \\
&= \sum_{u<v<w} (-1)^{u+v+w-1} \begin{vmatrix} \mathbf{a}_1 \cdot \mathbf{b}_u & \mathbf{a}_1 \cdot \mathbf{b}_v & \mathbf{a}_1 \cdot \mathbf{b}_w \\ \mathbf{a}_2 \cdot \mathbf{b}_u & \mathbf{a}_2 \cdot \mathbf{b}_v & \mathbf{a}_2 \cdot \mathbf{b}_w \\ \mathbf{a}_3 \cdot \mathbf{b}_u & \mathbf{a}_3 \cdot \mathbf{b}_v & \mathbf{a}_3 \cdot \mathbf{b}_w \end{vmatrix} (\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_u \cdots \check{\mathbf{b}}_v \cdots \check{\mathbf{b}}_w \cdots \wedge \mathbf{b}_k)
\end{aligned} \tag{9}$$

In particular for $k = 3$ we have

$$\begin{aligned}
&(\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \mathbf{b}_3) \\
&= - \begin{vmatrix} \mathbf{a}_1 \cdot \mathbf{b}_1 & \mathbf{a}_1 \cdot \mathbf{b}_2 & \mathbf{a}_1 \cdot \mathbf{b}_3 \\ \mathbf{a}_2 \cdot \mathbf{b}_1 & \mathbf{a}_2 \cdot \mathbf{b}_2 & \mathbf{a}_2 \cdot \mathbf{b}_3 \\ \mathbf{a}_3 \cdot \mathbf{b}_1 & \mathbf{a}_3 \cdot \mathbf{b}_2 & \mathbf{a}_3 \cdot \mathbf{b}_3 \end{vmatrix} = - \det[\mathbf{a}_i \cdot \mathbf{b}_j]
\end{aligned} \tag{10}$$

This can be substituted back into equation 9 to put it in a non determinant form.

$$
\begin{aligned}
&(\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \cdots \wedge \mathbf{b}_k) \\
&= \sum_{u<v<w} (-1)^{u+v+w} (\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_u \wedge \mathbf{b}_v \wedge \mathbf{b}_w)(\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_u \cdots \check{\mathbf{b}}_v \cdots \check{\mathbf{b}}_w \cdots \wedge \mathbf{b}_k)
\end{aligned} \tag{11}
$$

4 Induction on the result.

It is pretty clear that recursively performing these calculations will yield similar determinant and inner dot product reduction results.

4.1 dot product of like grade terms as determinant.

Let's consider the equal grade case first, summarizing the results so far

$$
\begin{aligned}
\mathbf{a} \cdot \mathbf{b} &= \det[\mathbf{a} \cdot \mathbf{b}] \\
(\mathbf{a}_1 \wedge \mathbf{a}_2) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2) &= -\det[\mathbf{a}_i \cdot \mathbf{b}_j] \\
(\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \mathbf{b}_3) &= -\det[\mathbf{a}_i \cdot \mathbf{b}_j]
\end{aligned}
$$

What will the sign be for the higher grade equivalents? It has the appearance of being related to the sign associated with blade reversion. To verify this calculate the dot product of a blade formed from a set of perpendicular unit vectors with itself.

$$
\begin{aligned}
&(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_k) \cdot (\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_k) \\
&\qquad = (-1)^{k(k-1)/2} (\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_k) \cdot (\mathbf{e}_k \wedge \cdots \wedge \mathbf{e}_2 \wedge \mathbf{e}_1) \\
&\qquad = (-1)^{k(k-1)/2} \mathbf{e}_1 \cdot (\mathbf{e}_2 \cdots (\mathbf{e}_k \cdot (\mathbf{e}_k \wedge \cdots \wedge \mathbf{e}_2 \wedge \mathbf{e}_1))) \\
&\qquad = (-1)^{k(k-1)/2} \mathbf{e}_1 \cdot (\mathbf{e}_2 \cdots (\mathbf{e}_{k-1} \cdot (\mathbf{e}_{k-1} \wedge \cdots \wedge \mathbf{e}_2 \wedge \mathbf{e}_1))) \\
&\qquad = \cdots \\
&\qquad = (-1)^{k(k-1)/2}
\end{aligned}
$$

This fixes the sign, and provides the induction hypothesis for the general case:

$$
(\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \cdots \wedge \mathbf{b}_k) = (-1)^{k(k-1)/2} \det[\mathbf{a}_i \cdot \mathbf{b}_j] \tag{12}
$$

Alternately, one can remove the sign change coefficient with reversion of one of the blades:

$$
(\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k) \cdot (\mathbf{b}_k \wedge \mathbf{b}_{k-1} \wedge \cdots \wedge \mathbf{b}_1) = \det[\mathbf{a}_i \cdot \mathbf{b}_j] \tag{13}
$$

4.2 Unlike grades.

Let's summarize the results for unlike grades at the same time reformulating the previous results in terms of index permuation, also writing for brevity $\mathbf{A}_s = \mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_s$, and $\mathbf{B}_k = \mathbf{b}_1 \wedge \cdots \wedge \mathbf{b}_k$:

$$\mathbf{A}_1 \cdot \mathbf{B}_k = \sum_i \operatorname{sgn}(\pi(i, 1, 2, \cdots \check{i} \cdots, k))(\mathbf{A}_1 \cdot \mathbf{b}_i)(\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_i \cdots \wedge \mathbf{b}_k)$$

$$\mathbf{A}_2 \cdot \mathbf{B}_k = \sum_{i_1 < i_2} \operatorname{sgn}(\pi(i_1, i_2, 1, 2, \cdots \check{i_1} \cdots \check{i_2} \cdots, k))$$
$$\mathbf{A}_2 \cdot (\mathbf{b}_{i_1} \wedge \mathbf{b}_{i_2})(\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_{i_1} \cdots \check{\mathbf{b}}_{i_2} \cdots \wedge \mathbf{b}_k)$$

$$\mathbf{A}_3 \cdot \mathbf{B}_k = \sum_{i_1 < i_2 < i_3} \operatorname{sgn}(\pi(i_1, i_2, i_3, 1, 2, \cdots \check{i_1} \cdots \check{i_2} \cdots \check{i_3} \cdots, k))$$
$$\mathbf{A}_3 \cdot (\mathbf{b}_{i_1} \wedge \mathbf{b}_{i_2} \wedge \mathbf{b}_{i_3})(\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_{i_1} \cdots \check{\mathbf{b}}_{i_2} \cdots \check{\mathbf{b}}_{i_3} \cdots \wedge \mathbf{b}_k)$$

We see that the dot product consumes any of the excess sign variation not described by the sign of the permutation of indexes.

The induction hypothysis is basically described above (change 3 to s, and add extra dots):

$$\mathbf{A}_s \cdot \mathbf{B}_k = \sum_{i_1 < i_2 \cdots < i_s} \operatorname{sgn}(\pi(i_1, i_2 \cdots, i_s, 1, 2, \cdots \check{i_1} \cdots \check{i_2} \cdots \check{i_s} \cdots, k))$$
$$\mathbf{A}_s \cdot (\mathbf{b}_{i_1} \wedge \mathbf{b}_{i_2} \cdots \wedge \mathbf{b}_{i_s})(\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_{i_1} \cdots \check{\mathbf{b}}_{i_2} \cdots \check{\mathbf{b}}_{i_s} \cdots \wedge \mathbf{b}_k) \tag{14}$$

4.3 Perform the induction.

In a sense this has already been done. The steps will be pretty much the same as the logic that produced the bivector and trivector results. Thinking about typing this up in latex isn't fun, so this will be left for a paper proof.