

Lorentz Invariance of Maxwell Lagrangians.

Peeter Joot

October 19, 2008. Last Revision: *Date* : 2008/11/1504 : 40 : 18

Contents

1 Working in multivector form.	1
1.1 Application of Lorentz boost to the field Lagrangian.	1
1.1.1 Maxwell equation invariance.	3
1.2 Lorentz boost applied to the Lorentz force Lagrangian.	3
2 Repeat in tensor form.	4
2.1 Translating versors to matrix form.	4
2.1.1 Revisit the expansion to matrix form above.	5
3 Translating versors tensor form.	5
3.1 Expressing vector Lorentz transform in tensor form.	6
3.2 Misc notes.	7
3.3 Expressing bivector Lorentz transform in tensor form.	7

1 Working in multivector form.

1.1 Application of Lorentz boost to the field Lagrangian.

The multivector form of the field Lagrangian is

$$\mathcal{L} = \kappa(\nabla \wedge A)^2 + A \cdot J \quad (1)$$

$$\kappa = -\frac{\epsilon_0 c}{2} \quad (2)$$

Write the boosting transformation on a four vector in exponential form

$$L(X) = \exp(\alpha \hat{\mathbf{a}}/2) X \exp(-\alpha \hat{\mathbf{a}}/2) = \Lambda X \Lambda^\dagger$$

where $\hat{\mathbf{a}} = a^i \gamma_i \wedge \gamma_0$ is any unit spacetime bivector, and α represents the rapidity angle.

Consider first the transformation of the interaction term with $A' = L(A)$, and $J' = L(J)$

$$\begin{aligned}
A' \cdot J' &= \langle L(A)L(J) \rangle \\
&= \langle \Lambda A \Lambda^\dagger \Lambda J \Lambda^\dagger \rangle \\
&= \langle \Lambda A J \Lambda^\dagger \rangle \\
&= \langle \Lambda^\dagger \Lambda A J \rangle \\
&= \langle A J \rangle \\
&= A \cdot J
\end{aligned}$$

Now consider the boost applied to the field bivector $F = \mathbf{E} + Ic\mathbf{B} = \nabla \wedge A$, by boosting both the gradient and the potential

$$\begin{aligned}
\nabla' \wedge A' &= L(\nabla) \wedge L(A) \\
&= \Lambda \nabla \wedge L(A) \\
&= (\Lambda \nabla \Lambda^\dagger) \wedge (\Lambda A \Lambda^\dagger) \\
&= \frac{1}{2} \left((\Lambda \nabla \Lambda^\dagger)(\Lambda A \Lambda^\dagger) - (\Lambda A \Lambda^\dagger)(\Lambda \nabla \Lambda^\dagger) \right) \\
&= \frac{1}{2} \left(\Lambda \nabla A \Lambda^\dagger - \Lambda A \nabla \Lambda^\dagger \right) \\
&= \Lambda (\nabla \wedge A) \Lambda^\dagger
\end{aligned}$$

The boosted squared field bivector in the Lagrangian is thus

$$\begin{aligned}
(\nabla' \wedge A')^2 &= \Lambda (\nabla \wedge A)^2 \Lambda^\dagger \\
&= \Lambda (\mathbf{E} + Ic\mathbf{B})^2 \Lambda^\dagger \\
&= \Lambda ((\mathbf{E}^2 - c^2\mathbf{B}^2) + 2Ic\mathbf{E} \cdot \mathbf{B}) \Lambda^\dagger \\
&= ((\mathbf{E}^2 - c^2\mathbf{B}^2) \Lambda \Lambda^\dagger + 2(\Lambda I \Lambda^\dagger) c\mathbf{E} \cdot \mathbf{B}) \\
&= ((\mathbf{E}^2 - c^2\mathbf{B}^2) + 2I \Lambda \Lambda^\dagger c\mathbf{E} \cdot \mathbf{B}) \\
&= ((\mathbf{E}^2 - c^2\mathbf{B}^2) + 2Ic\mathbf{E} \cdot \mathbf{B}) \\
&= (\mathbf{E} + Ic\mathbf{B})^2 \\
&= (\nabla \wedge A)^2
\end{aligned}$$

The commutation of the pseudoscalar I with the boost exponential $\Lambda = \exp(\alpha \hat{\mathbf{a}}/2) = \cosh(\alpha/2) + \hat{\mathbf{a}} \sinh(\alpha/2)$ is possible since I anticommutes with

all four vectors and thus commutes with bivectors, such as $\hat{\mathbf{a}}$. I also necessarily commutes with the scalar components of this exponential, and thus commutes with any even grade multivector.

Putting all the pieces together this shows that the Lagrangian in its entirety is a Lorentz invariant

$$\mathcal{L}' = \kappa(\nabla' \wedge A')^2 + A' \cdot J' = \kappa(\nabla \wedge A)^2 + A \cdot J = \mathcal{L}$$

FIXME: what is the conserved quantity associated with this? There should be one according to Noether's theorem? Is it the guage condition $\nabla \cdot A = 0$?

1.1.1 Maxwell equation invariance.

Somewhat related, having calculated the Lorentz transform of $F = \nabla \wedge A$, is an aside showing that the Maxwell equation is unsurprisingly also is a Lorentz invariant.

$$\begin{aligned}\nabla'(\nabla' \wedge A') &= J' \\ \Lambda \nabla \Lambda^\dagger \Lambda (\nabla \wedge A) \Lambda^\dagger &= \Lambda J \Lambda^\dagger \\ \Lambda \nabla (\nabla \wedge A) \Lambda^\dagger &= \Lambda J \Lambda^\dagger\end{aligned}$$

Pre and post multiplying with Λ^\dagger , and Λ respectively returns the unboosted equation

$$\nabla(\nabla \wedge A) = J$$

1.2 Lorentz boost applied to the Lorentz force Lagrangian.

Next interesting case is the Lorentz force, which for a time positive metric signature is:

$$\mathcal{L} = qA \cdot v/c + \frac{1}{2}mv \cdot v$$

The boost invariance of the $A \cdot J$ dot product demonstrated above demonstrates the general invariance property for any four vector dot product, and this Lagrangian has nothing but dot products in it. It thus follows directly that the Lorentz force Lagrangian is also a Lorentz invariant.

2 Repeat in tensor form.

Now, I can follow the above, but presented with the same sort of calculation in tensor form I am hopeless to understand it. To attempt translating this into tensor form, it appears the first step is putting the Lorentz transform itself into tensor or matrix form.

2.1 Translating versors to matrix form.

To get the feeling for how this will work, assume $\hat{\mathbf{a}} = \sigma_1$, so that the boost is along the x-axis. In that case we have

$$L(X) = (\cosh(\alpha/2) + \gamma_{10} \sinh(\alpha/2))x^\mu \gamma_\mu (\cosh(\alpha/2) + \gamma_{01} \sinh(\alpha/2))$$

Writing $C = \cosh(\alpha/2)$, and $S = \sinh(\alpha/2)$, and observing that the exponentials commute with the γ_2 , and γ_3 directions so the exponential action on those directions cancel.

$$L(X) = x^2 \gamma_2 + x^3 \gamma_3 + (C + \gamma_{10} S)(x^0 \gamma_0 + x^1 \gamma_1)(C + \gamma_{01} S)$$

Expanding just the non-perpendicular parts of the above

$$\begin{aligned} & (C + \gamma_{10} S)(x^0 \gamma_0 + x^1 \gamma_1)(C + \gamma_{01} S) \\ &= x^0 (C^2 \gamma_0 + \gamma_{10001} S^2) + x^0 S C (\gamma_{001} + \gamma_{100}) + x^1 (C^2 \gamma_1 + \gamma_{10101} S^2) + x^1 S C (\gamma_{101} + \gamma_{101}) \\ &= x^0 (C^2 \gamma_0 - \gamma_{01100} S^2) + 2x^0 S C \gamma_{001} + x^1 (C^2 \gamma_1 - \gamma_{11001} S^2) - 2x^1 S C \gamma_{011} \\ &= (x^0 \gamma_0 + x^1 \gamma_1)(C^2 + S^2) + 2(\gamma_0)^2 S C (x^0 \gamma_1 + x^1 \gamma_0) \\ &= (x^0 \gamma_0 + x^1 \gamma_1) \cosh(\alpha) + (\gamma_0)^2 \sinh(\alpha) (x^0 \gamma_1 + x^1 \gamma_0) \\ &= \gamma_0 (x^0 \cosh(\alpha) + x^1 \sinh((\gamma_0)^2 \alpha)) + \gamma_1 (x^1 \cosh(\alpha) + x^0 \sinh((\gamma_0)^2 \alpha)) \end{aligned}$$

In matrix form the complete transformation is thus

$$\begin{aligned} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}' &= \begin{bmatrix} \cosh(\alpha(\gamma_0)^2) & \sinh(\alpha(\gamma_0)^2) & 0 & 0 \\ \sinh(\alpha(\gamma_0)^2) & \cosh(\alpha(\gamma_0)^2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \\ &= \cosh(\alpha(\gamma_0)^2) \begin{bmatrix} 1 & \tanh(\alpha(\gamma_0)^2) & 0 & 0 \\ \tanh(\alpha(\gamma_0)^2) & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \end{aligned}$$

This supplies the specific meaning for the α factor in the exponential form, namely:

$$\begin{aligned}\alpha &= -\tanh^{-1}(\beta(\gamma_0)^2) \\ &= -\tanh^{-1}(|\mathbf{v}|/c(\gamma_0)^2)\end{aligned}$$

Or

$$\begin{aligned}\alpha \hat{\mathbf{a}} &= -\tanh^{-1}(\hat{\mathbf{a}}|\mathbf{v}|/c(\gamma_0)^2) \\ &= -\tanh^{-1}(\mathbf{v}/c(\gamma_0)^2)\end{aligned}$$

Putting this back into the original Lorentz boost equation to tidy it up, and writing $\tanh(\mathbf{A}) = \mathbf{v}/c$, the Lorentz boost is

$$L(X) = \begin{cases} \exp(-\mathbf{A}/2)X \exp(\mathbf{A}/2) & \text{for } (\gamma_0)^2 = 1 \\ \exp(\mathbf{A}/2)X \exp(-\mathbf{A}/2) & \text{for } (\gamma_0)^2 = -1 \end{cases}$$

Both of the metric signature options are indicated here for future reference and comparison with results using the alternate signature.

2.1.1 Revisit the expansion to matrix form above.

Looking back, multiplying out all the half angle terms as done above is this is the long dumb hard way to do it. A more sensible way would be to note that $\exp(\alpha \hat{\mathbf{a}}/2)$ anticommutes with both γ_0 and γ_1 thus

$$\begin{aligned}\exp(\alpha \hat{\mathbf{a}}/2)(x^0 \gamma_0 + x^1 \gamma_1) \exp(-\alpha \hat{\mathbf{a}}/2) &= \exp(\alpha \hat{\mathbf{a}})(x^0 \gamma_0 + x^1 \gamma_1) \\ &= (\cosh(\alpha) + \hat{\mathbf{a}} \sinh(\alpha))(x^0 \gamma_0 + x^1 \gamma_1)\end{aligned}$$

The matrix form thus follows directly.

3 Translating versors tensor form.

After this temporary digression back to the multivector form of the Lorentz transformation lets dispose of the specifics of the boost direction and magnitude, and also the metric signature. Instead encode all of these in a single versor variable Λ , again writing

$$L(X) = \Lambda X \Lambda^\dagger \tag{3}$$

3.1 Expressing vector Lorentz transform in tensor form.

What is the general way to encode this linear transformation in tensor/matrix form? The transformed vector is just that a vector, and thus can be written in terms of coordinates for some basis

$$\begin{aligned} L(X) &= (L(X) \cdot e^\mu) e_\mu \\ &= ((\Lambda(x^\nu \gamma_\nu) \Lambda^\dagger) \cdot e^\mu) e_\mu \\ &= x^\nu ((\Lambda \gamma_\nu \Lambda^\dagger) \cdot e^\mu) e_\mu \end{aligned}$$

The inner term is just the tensor that we want. Write

$$\begin{aligned} \Lambda_\nu{}^\mu &= (\Lambda \gamma_\nu \Lambda^\dagger) \cdot e^\mu \\ \Lambda^\nu{}_\mu &= (\Lambda \gamma^\nu \Lambda^\dagger) \cdot e_\mu \end{aligned}$$

for

$$\begin{aligned} L(X) &= x^\nu \Lambda_\nu{}^\mu e_\mu \\ &= x_\nu \Lambda^\nu{}_\mu e^\mu \end{aligned}$$

Completely eliminating the basis, working in just the coordinates $X = x'^\mu e_\mu = x'_\mu e^\mu$ this is

$$x'^\mu = x^\nu \Lambda_\nu{}^\mu \quad (4)$$

$$x'_\mu = x_\nu \Lambda^\nu{}_\mu \quad (5)$$

Now, in particular, having observed that the dot product is a Lorentz invariant this should supply the index manipulation rule for operating with the Lorentz boost tensor in a dot product context.

Write

$$\begin{aligned} L(X) \cdot L(Y) &= (x^\nu \Lambda_\nu{}^\mu e_\mu) \cdot (y_\alpha \Lambda^\alpha{}_\beta e^\beta) \\ &= x^\nu y_\alpha \Lambda_\nu{}^\mu \Lambda^\alpha{}_\beta e_\mu \cdot e^\beta \\ &= x^\nu y_\alpha \Lambda_\nu{}^\mu \Lambda^\alpha{}_\mu \end{aligned}$$

Since this equals $x^\nu y_\nu$, the tensor rule must therefore be

$$\Lambda_\mu{}^\sigma \Lambda^\nu{}_\sigma = \delta_\mu{}^\nu \quad (6)$$

After a somewhat long path, the core idea behind the Lorentz boost tensor is that it is the “matrix” of a linear transformation that leaves the four vector dot product unchanged. There is no need to consider any clifford algebra formulations to express just that idea.

3.2 Misc notes.

FIXME: To complete the expression of this in tensor form enumerating exactly how to express the dot product in tensor form would also be reasonable. ie: how to compute the reciprocal coordinates without describing the basis. Doing this will introduce the metric tensor into the mix.

Looks like the result 6 is consistent with [Minahan()] and that doc starts making a bit more sense now. I do see that he uses primes to distinguish the boost tensor from its inverse (using the inverse tensor (primed index down) to transform the covariant (down) coordinates). Is there a convention for keeping free vs. varied indexes close to the body of the operator? For the boost tensor he puts the free index closer to Λ , but for the inverse tensor for a covariant coordinate transformation puts the free index further out?

This also appears to be notational consistent with [Spence()].

3.3 Expressing bivector Lorentz transform in tensor form.

Having translated a vector Lorentz transform into tensor form, the next step is to do the same for a bivector. In particular for the field bivector $F = \nabla \wedge A$.

Write

$$\begin{aligned}\nabla' &= \Lambda \gamma_\mu \partial^\mu \Lambda^\dagger \\ A' &= \Lambda A^\nu \gamma_\nu \Lambda^\dagger\end{aligned}$$

$$\begin{aligned}\nabla' \cdot e^\beta &= (\Lambda \gamma_\mu \Lambda^\dagger) \cdot e^\beta \partial^\mu = \Lambda_\mu^\beta \partial^\mu \\ A' \cdot e^\beta &= (\Lambda \gamma_\nu \Lambda^\dagger) \cdot e^\beta A^\nu = \Lambda_\nu^\beta A^\nu\end{aligned}$$

Then the transformed bivector is

$$\begin{aligned}F' &= \nabla' \wedge A' = ((\nabla' \cdot e^\alpha) e_\alpha) \wedge ((A' \cdot e^\beta) e_\beta) \\ &= (e_\alpha \wedge e_\beta) \Lambda_\mu^\alpha \Lambda_\nu^\beta \partial^\mu A^\nu\end{aligned}$$

and finally the transformed tensor is thus

$$\begin{aligned}
F^{ab'} &= F' \cdot (e^b \wedge e^a) \\
&= (e_\alpha \wedge e_\beta) \cdot (e^b \wedge e^a) \Lambda_\mu^\alpha \Lambda_\nu^\beta \partial^\mu A^\nu \\
&= (\delta_\alpha^a \delta_\beta^b - \delta_\beta^a \delta_\alpha^b) \Lambda_\mu^\alpha \Lambda_\nu^\beta \partial^\mu A^\nu \\
&= \Lambda_\mu^a \Lambda_\nu^b \partial^\mu A^\nu - \Lambda_\mu^b \Lambda_\nu^a \partial^\mu A^\nu \\
&= \Lambda_\mu^a \Lambda_\nu^b (\partial^\mu A^\nu - \partial^\nu A^\mu)
\end{aligned}$$

Which gives the final transformation rule for the field bivector in tensor form

$$F^{ab'} = \Lambda_\mu^a \Lambda_\nu^b F^{\mu\nu} \quad (7)$$

Returning to the original problem of field Lagrangian invariance, we want to examine how $F^{ab'} F_{ab}'$ transforms. That is

$$F^{ab'} F_{ab}' = \Lambda_\mu^a \Lambda_\nu^b F^{\mu\nu} \Lambda_\alpha^a \Lambda_\beta^b F_{\alpha\beta} \quad (8)$$

$$= (\Lambda_\mu^a \Lambda_\alpha^a) (\Lambda_\nu^b \Lambda_\beta^b) F^{\mu\nu} F_{\alpha\beta} \quad (9)$$

$$= \delta_\mu^\alpha \delta_\nu^\beta F^{\mu\nu} F_{\alpha\beta} \quad (10)$$

$$= F^{\mu\nu} F_{\mu\nu} \quad (11)$$

which is the desired result. Since the dot product remainder of the Lagrangian 1 has already been shown to be Lorentz invariant this is sufficient to prove the Lagrangian boost or rotational invariance using tensor algebra.

Working this way is fairly compact and efficient, and required a few less steps than the multivector equivalent. To compare apples to apples, for the algebraic tools, it should be noted that if only the scalar part of $(\nabla \wedge A)^2$ was considered as implicitly done in the tensor argument above, the multivector approach would likely have been as compact as well.

References

[Minahan()] Joseph A. Minahan. Tensors without tears. "<http://www.teorfys.uu.se/people/minahan/Courses/SR/tensors.pdf>".

[Spence()] Prof. WJ Spence. Special relativity, four vectors, .. covariant and contravariant vectors, tensors. "<http://monopole.ph.qmw.ac.uk/~bill/emt/EMT7new.pdf>".