A couple small notes on canonican momentum.

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1

Goldstein defines the canonical momentum as:

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}$$

and gives an example (Lorentz force) about how this can generalize the concept of momentum to include contributions from velocity dependent potentials.

Lets look at his example, but put into the more natural covariant form with the Lorentz Lagrangian (using summation convention here)

$$\mathcal{L} = \frac{1}{2} m v^2 + q A \cdot v / c = \frac{1}{2} m \gamma_\alpha \cdot \gamma_\beta \dot{x}^\alpha \dot{x}^\beta + \frac{q}{c} \gamma_\alpha \cdot \gamma_\beta A^\alpha \dot{x}^\beta$$

Calculation of the canonical momentum gives:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} &= m \gamma_\alpha \cdot \gamma_\beta {\delta^\alpha}_\mu \dot{x}^\beta + \frac{q}{c} \gamma_\alpha \cdot \gamma_\beta A^\alpha {\delta^\beta}_\mu \\
&= m \gamma_\mu \cdot \gamma_\alpha \dot{x}^\alpha + \frac{q}{c} \gamma_\alpha \cdot \gamma_\mu A^\alpha \\
&= \gamma_\mu \cdot \left(m \gamma_\alpha \dot{x}^\alpha + \frac{q}{c} \gamma_\alpha A^\alpha \right) \\
&= \gamma_\mu \cdot \left(m v + \frac{q}{c} A \right)
\end{aligned}$$

So, if we are to call this modified quantity $p = mv + qA/c$ the total general momentum for the system, then the canonican momentum conjugate to x^μ is:

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \gamma_\mu \cdot p.$$

In terms of our reciprocal frame vectors, the components of p are:

$$p = \gamma_\mu \gamma^\mu \cdot p = \gamma_\mu p^\mu$$
$$p = \gamma^\mu \gamma_\mu \cdot p = \gamma^\mu p_\mu$$

From this we see that the conjugate momentum gives us our vector momentum component with respect to the reciprocal frame. We can therefore recover our total momentum by summing over the reciprocal frame vectors.

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x^\mu} &= \frac{d}{d\tau} \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \\
&= \frac{d}{d\tau} p_\mu \\
\implies \\
\sum \gamma^\mu \frac{\partial \mathcal{L}}{\partial x^\mu} &= \sum \frac{d}{d\tau} \gamma^\mu p_\mu
\end{aligned}$$

Observe that we have nothing more than our spacetime gradient on the left hand side, and a velocity specific spacetime gradient on the right hand side. Summarizing, this allows for writing the Euler-Lagrange equations in vector form as follows:

$$\frac{dp}{d\tau} = \nabla \mathcal{L} \tag{1}$$
$$p = \nabla_v \mathcal{L} \tag{2}$$
$$\nabla = \gamma^\mu \frac{\partial}{\partial x^\mu} \tag{3}$$
$$\nabla_v = \gamma^\mu \frac{\partial}{\partial \dot{x}^\mu} \tag{4}$$

Now, perhaps this is a step backwards, since the Lagrangian formulation allows for not having to use vector representations explicitly, nor to be constrained to specific parameterizations such as this constant frame vector representation. However, it is nice to see things in a form that is closer to what one is used to, and this isn't too different seeming than the familiar spatial Newtonian formulation:

$$\frac{d\mathbf{p}}{dt} = -\boldsymbol{\nabla} \phi$$