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1 Relativistic dynamics from Lagrangian.

David Tong's problem 3 on Lagrangian's is to show that the following

$$S = \kappa\sqrt{1 - \mathbf{v}^2/c^2} - \varphi \tag{1}$$

can be used to find the equation of motion. The constant κ is determined by requiring correspondence with the classical limit for small velocities:

$$\kappa\sqrt{1 - \mathbf{v}^2/c^2} \simeq \kappa\left(1 - \frac{1}{2}\mathbf{v}^2/c^2\right) = \text{constant} + \frac{1}{2}m\mathbf{v}^2 + \cdots$$

Thus by neglecting the higher order terms, we have the Newtonian kinetic energy for $\kappa = -mc^2$, and the action to minimize is:

$$S = -mc^2\sqrt{1 - \mathbf{v}^2/c^2} - \varphi \simeq \frac{1}{2}m\mathbf{v}^2 - \left(\varphi + mc^2\right) \tag{2}$$

1.1 Relativistic three vector equation of motion.

Performing the calculations

$$\begin{aligned}
\frac{\partial S}{\partial x^i} &= \frac{d}{dt}\frac{\partial S}{\partial \dot{x}^i} \\
-\frac{\partial \varphi}{\partial x^i} &= \frac{d}{dt}\left(-mc^2 \underbrace{\frac{1}{\sqrt{1 - \mathbf{v}^2/c^2}}}_{=\gamma}(\frac{1}{2})(-2\dot{x}^i/c^2)\right) \\
-\frac{\partial \varphi}{\partial x^i} &= \frac{d}{dt}\left(m\gamma\dot{x}^i\right) \\
\sum \mathbf{e}_i(-\frac{\partial \varphi}{\partial x^i}) &= \sum \mathbf{e}_i\frac{d}{dt}\left(m\gamma\dot{x}^i\right)
\end{aligned}$$

This provides the expected relativistically corrected equation of Newton's law:

$$\frac{d(\gamma \mathbf{p})}{dt} = -\nabla \varphi \tag{3}$$

1.2 Relativistic four vector equation of motion.

Logically the above isn't satisfactory. We want a four vector version of it. Can the same Lagrangian be used to achieve this. I thought I had made such a calculation with what I guessed was the appropriate modification of the Lagrangian equations to treat space and time symmetrically:

$$\frac{\partial S}{\partial x^\mu} = \frac{d}{ds}\left(\frac{\partial S}{\partial \frac{dx^\mu}{ds}}\right) \tag{4}$$

I have no proof that this is valid, and really need to study some variational calculus before I can make any such claim. I've also been unable to satisfactorily reproduce my original derivation of:

$$F = \frac{d}{d\tau}\left(m\frac{dX}{d\tau}\right) \tag{5}$$

I may have made compensatory errors to arrive at the answer I desired. Have to go dig up my original notes where I thought I had done this.

2 OLDER FOUR VECTOR NOTES.

I was summarizing for myself the various four-vectors of mechanics:

$$\begin{aligned}
X &= ct + \mathbf{X} \\
V &= \frac{dX}{d\tau} = \gamma(c + \mathbf{v}) \\
P &= mV = E/c + \gamma\mathbf{p} \\
f &= m\frac{d^2X}{d\tau^2} = m\frac{dV}{d\tau}
\end{aligned}$$

where:

$$
\begin{aligned}
\gamma^{-2} &= 1 - |\mathbf{v}/c|^2 \\
ds = c d\tau &= \left(\frac{dx}{d\lambda} \cdot \frac{dx}{d\lambda} \right)^{1/2} d\lambda \\
X \cdot X = |X|^2 &= c^2 t^2 - |\mathbf{X}|^2 \\
E &= \int f \cdot (c d\tau) \\
\mathbf{v} &= \frac{d\mathbf{X}}{dt} \\
\mathbf{p} &= m\mathbf{v}
\end{aligned}
$$

Invarients for the first three four vectors are:

$$
\begin{aligned}
|X|^2 &= c^2 t^2 - |\mathbf{X}|^2 = c^2 \tau^2 \\
|V|^2 &= \gamma^2 (c^2 - |\mathbf{v}|^2) = c^2 \\
|P|^2 &= m^2 |V|^2 = m^2 c^2
\end{aligned}
$$

Is the Minkowski norm of the four vector force:

$$
f = m \frac{d^2 X}{d\tau^2}
$$

also an invarient? I think it has to be. Assuming that is the case, what would the value (and significance if any) of this be?