Some notes on GAFP 5.5.3 The Lorentz force Law.

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The idea behind this derivation, is to express the vector part of the proper force in covarient form, and then do the same for the energy change part of the proper momentum. That first part is:

$$\begin{aligned}
\frac{dp}{d\tau} \wedge \gamma_0 &= \frac{d(\gamma \mathbf{p})}{d\tau} \\
&= \frac{d(\gamma \mathbf{p})}{dt} \frac{dt}{d\tau} \\
&= \frac{dt}{d\tau} q \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right)
\end{aligned}$$

Now, the spacetime split of velocity is done in the normal fashion:

$$\begin{aligned}
x &= ct\gamma_0 + \sum x^i \gamma_i \\
v &= \frac{dx}{d\tau} = c\frac{dt}{d\tau}\gamma_0 + \sum \frac{dx^i}{d\tau}\gamma_i \\
v \cdot \gamma_0 &= c\frac{dt}{d\tau} = c\gamma \\
v \wedge \gamma_0 &= \sum \frac{dx^i}{dt}\frac{dt}{d\tau}\gamma_i\gamma_0 \\
&= (v \cdot \gamma_0)/c \sum v^i \sigma_i \\
&= (v \cdot \gamma_0)\mathbf{v}/c.
\end{aligned}$$

Writing $\dot{p} = dp/d\tau$, substituite the gamma factor into the force equation:

$$\dot{p} \wedge \gamma_0 = (v/c \cdot \gamma_0) q \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right)$$

Now, GAFP goes on to show that the $\gamma \mathbf{E}$ term can be reduced to the form $(\mathbf{E} \cdot v) \wedge \gamma_0$. Their method isn't exactly obvious, for example writing $\mathbf{E} =$

$(1/2)(\mathbf{E} + \mathbf{E})$ to start. Let's just do this backwards instead, expanding $\mathbf{E} \cdot v$ to see the form of that term:

$$
\begin{aligned}
\mathbf{E} \cdot v &= \left(\sum E^i \gamma_{i0}\right) \cdot \left(\sum v^\mu \gamma_\mu\right) \\
&= \sum E^i v^\mu \left\langle \gamma_{i0\mu} \right\rangle_1 \\
&= v^0 \sum E^i \gamma_i + \sum E^i v^j \underbrace{\left\langle \gamma_{i0j} \right\rangle_1}_{-\delta_{ij}\gamma_0} \\
&= v^0 \sum E^i \gamma_i - \sum E^i v^i \gamma_0.
\end{aligned}
$$

Wedging with γ_0 we have the desired result:

$$
(\mathbf{E} \cdot v) \wedge \gamma_0 = v^0 \sum E^i \gamma_{i0} = (v \cdot \gamma_0)\mathbf{E} = c\gamma\mathbf{E}
$$

Now, for equation 5.164 there aren't any suprising steps, but lets try this backwards too:

$$
\begin{aligned}
(I\mathbf{B}) \cdot v &= \left(\sum B^i \underbrace{\gamma_{102030i0}}_{\gamma_{123i}}\right) \cdot \left(\sum v^\mu \gamma_\mu\right) \\
&= \sum B^i v^\mu \left\langle \gamma_{123i\mu} \right\rangle_1
\end{aligned}
$$

That vector selection does yield the cross product as expected:

$$
\left\langle \gamma_{123i\mu} \right\rangle_1 = \begin{cases}
0 & \mu = 0 \\
0 & i = \mu \\
\gamma_1 & i\mu = 32 \\
-\gamma_2 & i\mu = 31 \\
\gamma_3 & i\mu = 21
\end{cases}
$$

(with alternation for the missing set of index pairs).
This gives:

$$
(I\mathbf{B}) \cdot v = (B^3 v^2 - B^2 v^3)\gamma_1 + (B^1 v^3 - B^3 v^1)\gamma_2 + (B^2 v^1 - B^1 v^2)\gamma_3,
$$

thus, since $v^i = \gamma dx^i/dt$, this yields the desired result

$$
((I\mathbf{B}) \cdot v) \wedge \gamma_0 = \gamma \mathbf{v} \times \mathbf{B}
$$

In retrospect, for this magnetic field term, the GAFP approach is cleaner and easier than to try to do it the dumb way.

Combining the results we have:

$$\begin{aligned}
\dot{p} \wedge \gamma_0 &= q\gamma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \\
&= q((\mathbf{E} + cI\mathbf{B}) \cdot (v/c)) \wedge \gamma_0
\end{aligned}$$

Or with $F = \mathbf{E} + cI\mathbf{B}$, we have:

$$\dot{p} \wedge \gamma_0 = q(F \cdot v/c) \wedge \gamma_0 \tag{1}$$

It is tempting here to attempt to cancel the $\wedge\gamma_0$ parts of this equation, but that cannot be done until one also shows:

$$\dot{p} \cdot \gamma_0 = q(F \cdot v/c) \cdot \gamma_0$$

I follow most of the details of GAFP on this fine. I found they omitted a couple steps that would have been helpful.

For the four momentum we have:

$$p_0 = p \cdot \gamma_0 = E/c$$

The rate of change work done on the particle by the force is:

$$\begin{aligned}
dW &= q\mathbf{E} \cdot d\mathbf{x} \\
\frac{dW}{dt} &= q\mathbf{E} \cdot \frac{d\mathbf{x}}{dt} = c\frac{dp_0}{dt} \\
\frac{dp_0}{dt} &= q\mathbf{E} \cdot \mathbf{v}/c \\
\frac{dp_0}{d\tau} &= \underbrace{\frac{dt}{d\tau}}_{v/c\cdot\gamma_0} q\mathbf{E} \cdot \left(\frac{v \wedge \gamma_0}{v \cdot \gamma_0}\right) \\
&= q\mathbf{E} \cdot (v/c \wedge \gamma_0) \\
&= q\left(\mathbf{E} + cI\mathbf{B}\right) \cdot (v/c \wedge \gamma_0)
\end{aligned}$$

$I\mathbf{B}$ has only purely spatial bivectors, γ_{12}, γ_{13}, and γ_{23}. On the other hand $v \wedge \gamma_0 = \sum v^i \gamma_{i0}$ has only spacetime bivectors, so $I\mathbf{B} \cdot (v/c \wedge \gamma_0) = 0$, which is why it can be added above to complete the field.

That leaves:

$$\frac{dp_0}{d\tau} = qF \cdot (v/c \wedge \gamma_0)\,, \tag{2}$$

but we want to put this in the same form as equation 1. To do so, note how we can reduce the dot product of two bivectors:

$$\begin{aligned}
(a \wedge b) \cdot (c \wedge d) &= \langle (a \wedge b)(c \wedge d) \rangle \\
&= \langle (a \wedge b)(cd - c \cdot d) \rangle \\
&= \langle ((a \wedge b) \cdot c)d + ((a \wedge b) \wedge c)d \rangle \\
&= ((a \wedge b) \cdot c) \cdot d.
\end{aligned}$$

Using this, and adding the result to equation 1 we have:

$$\dot{p} \cdot \gamma_0 + \dot{p} \wedge \gamma_0 = q(F \cdot v/c) \cdot \gamma_0 + q(F \cdot v/c) \wedge \gamma_0$$

Or

$$\dot{p}\gamma_0 = q(F \cdot v/c)\gamma_0$$

Right multiplying by γ_0 on both sides to cancel those terms we have our end result, the covarient form of the Lorentz proper force equation:

$$\dot{p} = q(F \cdot v/c) \tag{3}$$

2 Lorentz force in terms of four potential.

If one expresses the Faraday bivector in terms of a spacetime curl of a potential vector:

$$F = \nabla \wedge A, \tag{4}$$

then inserting into equation 3 we have:

$$\begin{aligned}
\dot{p} &= q(F \cdot v/c) \\
&= q(\nabla \wedge A) \cdot v/c \\
&= q\left(\nabla(A \cdot v/c) - A(\nabla \cdot v/c)\right)
\end{aligned}$$

Let's look at that proper velocity divergence term:

$$\begin{aligned}
\nabla \cdot v/c &= \frac{1}{c}\left(\nabla \cdot \frac{dx}{d\tau}\right) \\
&= \frac{1}{c}\frac{d}{d\tau}\nabla \cdot x \\
&= \frac{1}{c}\frac{d}{d\tau}\sum \frac{\partial x^\mu}{\partial x^\mu} \\
&= \frac{1}{c}\frac{d4}{d\tau} \\
&= 0
\end{aligned}$$

This leaves the proper Lorentz force expressable as the (spacetime) gradient of a scalar quantity:

$$\dot{p} = q\nabla(A \cdot v/c) \tag{5}$$

I believe this dot product is likely an invariant of electromagnetism. Looking from the rest frame one has:

$$\dot{p} = q\nabla A^0 = q\sum \gamma^\mu \partial_\mu A^0 = \sum E^i \gamma_i \tag{6}$$

Wedging with γ_0 to calculate $\mathbf{E} = \sum E^i \gamma_i$, we have:

$$q\sum -\gamma_{i0}\partial_i A^0 = -q\boldsymbol{\nabla} A^0$$

So we want to identify this component of the four vector potential with electrostatic potential:

$$A^0 = \phi \tag{7}$$

3 Explicit expansion of potential spacetime curl in components.

Having used the gauge condition $\nabla \cdot A = 0$, to express the Faraday bivector as a gradient, we should be able to verify that this produces the familiar equations for $\mathbf{E}$, and $\mathbf{B}$ in terms of ϕ, and $\mathbf{A}$.

First lets do the electric field components, which are easier.

With $F = E + icB = \nabla \wedge A$, we calculate $\mathbf{E} = \sum \sigma_i E^i = \sum \gamma_{i0} E^i$.

$$\begin{aligned}
E^i &= F \cdot \left(\gamma^0 \wedge \gamma^i\right) = F \cdot \gamma^{0i} \\
&= \left(\sum \gamma^\mu \partial_\mu \wedge \gamma_\nu A^\nu\right) \cdot \gamma^{0i} \\
&= \sum \partial_\mu A^\nu {\gamma^\mu}_\nu \cdot \gamma^{0i} \\
&= \partial_0 A^i {\gamma^0}_i \cdot \gamma^{0i} + \partial_i A^0 {\gamma^i}_0 \cdot \gamma^{0i} \\
&= -\left(\partial_0 A^i + \partial_i A^0\right) \\
\sum E^i \sigma_i &= -\left(\partial_{ct} \sum \sigma_i A^i + \sum \sigma_i \partial_i A^0\right) \\
&= -\left(\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \boldsymbol{\nabla} A^0\right)
\end{aligned}$$

Again we see that we should identify $A^0 = \phi$, and write:

$$\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = -\boldsymbol{\nabla}\phi \tag{8}$$

Now, let's calculate the magnetic field components (setting $c = 1$ temporarily):

$$
\begin{aligned}
i\mathbf{B} &= \sigma_{123} \sum \sigma_i B^i \\
&= \sum \sigma_{123i} B^i \\
&= \sigma_{1231} B^1 + \sigma_{1232} B^2 + \sigma_{1233} B^3 \\
&= \sigma_{23} B^1 + \sigma_{31} B^2 + \sigma_{12} B^3 \\
&= \gamma_{2030} B^1 + \gamma_{3010} B^2 + \gamma_{1020} B^3 \\
&= \gamma_{32} B^1 + \gamma_{13} B^2 + \gamma_{21} B^3
\end{aligned}
$$

Thus, we can calculate the magnetic field components with:

$$B^1 = F \cdot \gamma^{23} \tag{9}$$

$$B^2 = F \cdot \gamma^{31} \tag{10}$$

$$B^3 = F \cdot \gamma^{12} \tag{11}$$

Here the components of F of interest are: $\gamma^i \wedge \gamma_j \partial_i A^j = -\gamma_{ij} \partial_i A^j$.

$$
\begin{aligned}
B^1 &= -\partial_2 A^3 \gamma_{23} \cdot \gamma^{23} - \partial_3 A^2 \gamma_{32} \cdot \gamma^{23} \\
B^2 &= -\partial_3 A^1 \gamma_{31} \cdot \gamma^{31} - \partial_1 A^3 \gamma_{13} \cdot \gamma^{31} \\
B^3 &= -\partial_1 A^2 \gamma_{12} \cdot \gamma^{12} - \partial_2 A^1 \gamma_{21} \cdot \gamma^{12} \\
\Longrightarrow & \\
B^1 &= \partial_2 A^3 - \partial_3 A^2 \\
B^2 &= \partial_3 A^1 - \partial_1 A^3 \\
B^3 &= \partial_1 A^2 - \partial_2 A^1
\end{aligned}
$$

Or, with $\mathbf{A} = \sum \sigma_i A^i$ and $\boldsymbol{\nabla} = \sum \sigma_i \partial_i$, this is our familiar:

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \tag{12}$$