NFCM Exercise 8.4. Legendre Polynomials

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This exercise is to find the first couple terms of the Legendre polynomial expansion of

$$\frac{1}{|\mathbf{x}-\mathbf{a}|}$$

Write

$$f(x) = \frac{1}{|\mathbf{x}|}$$

Expanding $f(\mathbf{x}-\mathbf{a})$ about $\mathbf{x}$ we have

$$\frac{1}{|\mathbf{x}-\mathbf{a}|} = \sum_{k=0} \frac{1}{k!}(-\mathbf{a}\cdot\nabla)^k \frac{1}{|\mathbf{x}|}$$

Expanding the first term we have

$$\begin{aligned}
-\mathbf{a}\cdot\nabla\frac{1}{|\mathbf{x}|} &= \frac{1}{|\mathbf{x}|^2}\mathbf{a}\cdot\nabla|\mathbf{x}| \\
&= \frac{1}{|\mathbf{x}|^2}\mathbf{a}\cdot\nabla(\mathbf{x}^2)^{1/2} \\
&= \frac{1}{|\mathbf{x}|^2}\frac{(1/2)}{(|\mathbf{x}|^2)^{1/2}}\mathbf{a}\cdot\nabla\mathbf{x}^2 \\
&= \frac{\mathbf{a}\cdot\mathbf{x}}{|\mathbf{x}|^3}
\end{aligned}$$

Expansion of the second derivative term is

$$\frac{(-\mathbf{a}\cdot\nabla)}{2}\frac{(-\mathbf{a}\cdot\nabla)}{1}\frac{1}{|\mathbf{x}|} = \frac{\mathbf{a}\cdot\nabla}{2}\left(\frac{-\mathbf{a}\cdot\mathbf{x}}{|\mathbf{x}|^3}\right)$$
$$= \frac{-1}{2}\left(\frac{\mathbf{a}\cdot\nabla(\mathbf{a}\cdot\mathbf{x})}{|\mathbf{x}|^3} + (\mathbf{a}\cdot\mathbf{x})\mathbf{a}\cdot\nabla\frac{1}{|\mathbf{x}|^3}\right)$$

For this we need

$$\mathbf{a}\cdot\nabla(\mathbf{a}\cdot\mathbf{x}) = \mathbf{a}\cdot(\mathbf{a}\cdot\nabla\mathbf{x}) = \mathbf{a}^2$$

And

$$\mathbf{a}\cdot\nabla\frac{1}{|\mathbf{x}|^k} = k\frac{1}{|\mathbf{x}|^{k-1}}\mathbf{a}\cdot\nabla\frac{1}{|\mathbf{x}|}$$
$$= k\frac{1}{|\mathbf{x}|^{k-1}}\frac{-\mathbf{a}\cdot\mathbf{x}}{|\mathbf{x}|^3}$$
$$= -k\frac{\mathbf{a}\cdot\mathbf{x}}{|\mathbf{x}|^{k+2}}$$

Thus the second derivative term is

$$\frac{-1}{2}\left(\frac{\mathbf{a}^2}{|\mathbf{x}|^3} - 3\frac{(\mathbf{a}\cdot\mathbf{x})^2}{|\mathbf{x}|^5}\right)$$
$$= \frac{(1/2)\left(3(\mathbf{a}\cdot\mathbf{x})^2 - \mathbf{a}^2\mathbf{x}^2\right)}{|\mathbf{x}|^5}$$

Summing these terms we have

$$\frac{1}{|\mathbf{x}-\mathbf{a}|} = \frac{1}{|\mathbf{x}|} + \frac{\mathbf{a}\cdot\mathbf{x}}{|\mathbf{x}|^3} + \frac{(1/2)\left(3(\mathbf{a}\cdot\mathbf{x})^2 - \mathbf{a}^2\mathbf{x}^2\right)}{|\mathbf{x}|^5} + \cdots$$

NFCM writes this as

$$\frac{1}{|\mathbf{x}-\mathbf{a}|} = \frac{P_0(\mathbf{x}\mathbf{a})}{|\mathbf{x}|} + \frac{P_1(\mathbf{x}\mathbf{a})}{|\mathbf{x}|^3} + \frac{P_2(\mathbf{x}\mathbf{a})}{|\mathbf{x}|^5} + \cdots$$

And calls $P_i = P_i(\mathbf{x}\mathbf{a})$ terms the Legendre polynomials. This isn't terribly clear since one expects a different form for the Legendre polynomials.

Using the Taylor formula one can derive a recurrance relation for these that makes the calculation a bit simpler

$$
\begin{aligned}
\frac{P_{k+1}}{|\mathbf{x}|^{2(k+1)+1}} &= \frac{-\mathbf{a}\cdot\nabla}{k+1}\left(\frac{P_k}{|\mathbf{x}|^{2k+1}}\right) \\
&= \frac{-1}{k+1}\left(\frac{\mathbf{a}\cdot\nabla(P_k}{|\mathbf{x}|^{2k+1}} + P_k\frac{\mathbf{a}\cdot\nabla}{|\mathbf{x}|^{2k+1}}\right) \\
&= \frac{1}{k+1}\left(P_k(2k+1)\frac{\mathbf{a}\cdot\mathbf{x}}{|\mathbf{x}|^{2k+3}} - \mathbf{x}^2\frac{\mathbf{a}\cdot\nabla P_k}{|\mathbf{x}|^{2k+3}}\right)
\end{aligned}
$$

Or

$$
(k+1)P_{k+1} = P_k(2k+1)\mathbf{a}\cdot\mathbf{x} - \mathbf{x}^2\mathbf{a}\cdot\nabla P_k
$$

Some of these have been calculated

$$
\begin{aligned}
P_0 &= 1 \\
P_1 &= \mathbf{a}\cdot\mathbf{x} \\
P_2 &= \frac{1}{2}(3(\mathbf{a}\cdot\mathbf{x})^2 - \mathbf{a}^2\mathbf{x}^2)
\end{aligned}
$$

And for the derivatives

$$
\begin{aligned}
\mathbf{a}\cdot\nabla P_0 &= 0 \\
\mathbf{a}\cdot\nabla P_1 &= \mathbf{a}^2 \\
\mathbf{a}\cdot\nabla P_2 &= \frac{1}{2}((3)(2)(\mathbf{a}\cdot\mathbf{x})\mathbf{a}^2 - 2\mathbf{a}^2\mathbf{x}\cdot\mathbf{a}) \\
&= 2\mathbf{a}^2(\mathbf{x}\cdot\mathbf{a})
\end{aligned}
$$

Using the recurrance relation one can calculate P_3 for example.

$$
\begin{aligned}
P_3 &= (1/3)\left(\frac{5}{2}(3(\mathbf{a}\cdot\mathbf{x})^2 - \mathbf{a}^2\mathbf{x}^2)(\mathbf{a}\cdot\mathbf{x}) - 2\mathbf{x}^2\mathbf{a}^2(\mathbf{x}\cdot\mathbf{a})\right) \\
&= (1/3)(\mathbf{a}\cdot\mathbf{x})\left(\frac{5}{2}(3(\mathbf{a}\cdot\mathbf{x})^2 - \mathbf{a}^2\mathbf{x}^2) - 2\mathbf{x}^2\mathbf{a}^2\right) \\
&= (\mathbf{a}\cdot\mathbf{x})\left(\frac{5}{2}((\mathbf{a}\cdot\mathbf{x})^2) - 3/2\mathbf{x}^2\mathbf{a}^2\right) \\
&= \frac{1}{2}(\mathbf{a}\cdot\mathbf{x})(5(\mathbf{a}\cdot\mathbf{x})^2 - 3\mathbf{x}^2\mathbf{a}^2)
\end{aligned}
$$

1.1 Putting things in standard Legendre polynomial form.

This is still pretty labourious to calculate, especially because of not having a closed form recurrance relation for $\mathbf{a} \cdot \nabla P_k$. Let's relate these to the standard Legendre polynomial form.

Observe that we can write

$$
\begin{aligned}
P_0(\mathbf{x}\mathbf{a}) &= 1 \\
\frac{P_1(\mathbf{x}\mathbf{a})}{|\mathbf{x}||\mathbf{a}|} &= \hat{\mathbf{a}} \cdot \hat{\mathbf{x}} \\
\frac{P_2(\mathbf{x}\mathbf{a})}{|\mathbf{x}|^2|\mathbf{a}|^2} &= \frac{1}{2}(3(\hat{\mathbf{a}} \cdot \hat{\mathbf{x}})^2 - 1) \\
\frac{P_3(\mathbf{x}\mathbf{a})}{|\mathbf{x}|^3|\mathbf{a}|^3} &= \frac{1}{2}(5(\hat{\mathbf{a}} \cdot \hat{\mathbf{x}})^3 - 3(\hat{\mathbf{a}} \cdot \hat{\mathbf{x}}))
\end{aligned}
$$

With this scaling, we have the standard form for the Legendre polynomials, and can write

$$
\frac{1}{\mathbf{x}-\mathbf{a}} = \frac{1}{|\mathbf{x}|}\left(P_0 + \frac{|\mathbf{a}|}{|\mathbf{x}|}P_1(\hat{\mathbf{a}} \cdot \hat{\mathbf{x}}) + \left(\frac{|\mathbf{a}|}{|\mathbf{x}|}\right)^2 P_2(\hat{\mathbf{a}} \cdot \hat{\mathbf{x}}) + \left(\frac{|\mathbf{a}|}{|\mathbf{x}|}\right)^3 P_3(\hat{\mathbf{a}} \cdot \hat{\mathbf{x}}) + \cdots\right)
$$

1.2 Scaling standard form Legendre polynomials

Since the odd Legendre polynomials have only odd terms and even have only even terms this allows for the scaled form that NFCM uses.

$$
\begin{aligned}
P_0(\mathbf{x}\mathbf{a}) &= P_0(\hat{\mathbf{a}} \cdot \hat{\mathbf{x}}) \\
P_1(\mathbf{x}\mathbf{a}) &= |\mathbf{x}||\mathbf{a}|P_1(\hat{\mathbf{a}} \cdot \hat{\mathbf{x}}) = \mathbf{a} \cdot \mathbf{x} \\
P_2(\mathbf{x}\mathbf{a}) &= |\mathbf{x}|^2|\mathbf{a}|^2 P_2(\hat{\mathbf{a}} \cdot \hat{\mathbf{x}}) = \frac{1}{2}(3(\mathbf{a} \cdot \mathbf{x})^2 - \mathbf{x}^2\mathbf{a}^2) \\
P_3(\mathbf{x}\mathbf{a}) &= |\mathbf{x}|^3|\mathbf{a}|^3 P_3(\hat{\mathbf{a}} \cdot \hat{\mathbf{x}}) = \frac{1}{2}(5(\mathbf{a} \cdot \mathbf{x})^3 - 3(\mathbf{a} \cdot \mathbf{x})\mathbf{x}^2\mathbf{a}^2)
\end{aligned}
$$

Every term for the k^{th} polynomial is a permutation of the geometric product $\mathbf{x}^k\mathbf{a}^k$.

This allows for writing some of these terms using the wedge product. Using the product expansion:

$$
(\mathbf{a} \cdot \mathbf{x})^2 = (\mathbf{a} \wedge \mathbf{x})^2 + \mathbf{a}^2\mathbf{x}^2
$$

Thus we have:

$$
\begin{aligned}
P_2(\mathbf{xa}) &= (\mathbf{a}\cdot\mathbf{x})^2 + \frac{1}{2}(\mathbf{a}\wedge\mathbf{x})^2 \\
&= (\mathbf{a}\cdot\mathbf{x})^2 - \frac{1}{2}|\mathbf{a}\wedge\mathbf{x}|^2
\end{aligned}
$$

This is nice geometrically since the directional dependence of this term on the colinearity and perpendicularity of the vectors **a** and **x** is clear.

Doing the same for the P_3:

$$
\begin{aligned}
P_3(\mathbf{xa}) &= (\mathbf{a}\cdot\mathbf{x})\frac{1}{2}(5(\mathbf{a}\cdot\mathbf{x})^2 - 3\mathbf{x}^2\mathbf{a}^2) \\
&= (\mathbf{a}\cdot\mathbf{x})\frac{1}{2}(2(\mathbf{a}\cdot\mathbf{x})^2 + 3(\mathbf{a}\wedge\mathbf{x})^2) \\
&= (\mathbf{a}\cdot\mathbf{x})((\mathbf{a}\cdot\mathbf{x})^2 - \frac{3}{2}|\mathbf{a}\wedge\mathbf{x}|^2)
\end{aligned}
$$

I suppose that one could get the same geometrical interpretation with a standard Legendre expansion in terms of $\hat{\mathbf{a}}\cdot\hat{\mathbf{x}} = cos(\theta)$ terms, by collect both $sin(\theta)$ and $cos(\theta)$ powers, but one can see the power of writing things explicitly in terms of the original vectors.

1.3 Note on NFCM Legendre polynomial notation.

In NFCM's slightly abusive notation P_k was used with various meanings. He wrote $P_k(\hat{\mathbf{a}}\cdot\hat{\mathbf{x}}) = \frac{P_k(\mathbf{xa})}{|\mathbf{x}|^k|\mathbf{a}|^k}$.

Note for example that the standard first degree Legendre polynomial $P_1(x) = x$ evaluated with a **xa** value:

$$
\begin{aligned}
\frac{1}{|\mathbf{x}||\mathbf{a}|}P_1(x)|_{x=\mathbf{xa}} &= \hat{\mathbf{x}}\hat{\mathbf{a}} \\
&= \hat{\mathbf{x}}\cdot\hat{\mathbf{a}} + \hat{\mathbf{x}}\wedge\hat{\mathbf{a}}
\end{aligned}
$$

This has a bivector component in addition to the component identical to the standard Legendre polynomial term (the first part).

By luck it happens that the scalar part of this equals $P_1(\hat{\mathbf{a}}\cdot\hat{\mathbf{x}})$, but this isn't the case for other terms. Example, $P_2(\mathbf{xa})$:

$$
\begin{aligned}
P_2(x)|_{x=\mathbf{xa}} &= \frac{1}{2}(3(\mathbf{xa})^2 - 1) \\
&= \frac{1}{2}(3(-\mathbf{ax} + 2\mathbf{a} \cdot \mathbf{x})(\mathbf{xa}) - 1) \\
&= \frac{1}{2}(3(-\mathbf{a}^2\mathbf{x}^2 + 2(\mathbf{a} \cdot \mathbf{x})^2 + 2(\mathbf{a} \cdot \mathbf{x})(\mathbf{x} \wedge \mathbf{a})) - 1) \\
&= -(3/2)\mathbf{a}^2\mathbf{x}^2 + 3(\mathbf{a} \cdot \mathbf{x})^2 + 3(\mathbf{a} \cdot \mathbf{x})(\mathbf{x} \wedge \mathbf{a}) - 1/2
\end{aligned}
$$

Scaling this by $1/(\mathbf{a}^2\mathbf{x}^2)$ is

$$
-\frac{3}{2} + 3(\hat{\mathbf{a}} \cdot \hat{\mathbf{x}})^2 + 3(\hat{\mathbf{a}} \cdot \hat{\mathbf{x}})(\hat{\mathbf{x}} \wedge \hat{\mathbf{a}}) - \frac{1}{\mathbf{a}^2\mathbf{x}^2}
$$

The scalar part of this isn't anything recognizable.