Revisit Lorentz force from Lagrangian.

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1 Motivation

In [Joot(b)] a derivation of the Lorentz force in covariant form was performed. Intuition says that result, because of the squared proper velocity, was dependent on the positive time Minkowski signature. With many GR references using the opposite signature, it seems worthwhile to understand what results are signature dependent and put them in a signature invariant form.

Here the result will be rederived without assuming this signature.

Assume a Lagrangian of the following form

$$\mathcal{L} = \frac{1}{2}mv^2 + \kappa A \cdot v \tag{1}$$

where v is the proper velocity. Here $A(x^\mu, \dot{x}^\nu) = A(x^\mu)$ is a position but not velocity dependent four vector potential. The constant κ includes the charge of the test mass, and will be determined exactly in due course.

2 Equations of motion.

As observed in [Joot(a)] the Euler-Lagrange equations can be summarized in four-vector form as

$$\nabla \mathcal{L} = \frac{d}{d\tau}(\nabla_v \mathcal{L}). \tag{2}$$

To compute this, some intermediate calculations are helpful

$$\begin{aligned}
\nabla v^2 &= 0 \\
\nabla(A \cdot v) &= \nabla A_\mu \dot{x}^\mu \\
&= \gamma^\nu \dot{x}^\mu \partial_\nu A_\mu \\
\frac{1}{2}\nabla_v v^2 &= \frac{1}{2}\nabla_v (\gamma_\mu)^2 (\dot{x}^\mu)^2 \\
&= \gamma^\nu (\gamma_\mu)^2 \partial_{\dot{x}^\nu} \dot{x}^\mu \\
&= \gamma^\mu (\gamma_\mu)^2 \dot{x}^\mu \\
&= \gamma_\mu \dot{x}^\mu \\
&= v \\
\nabla_v (A \cdot v) &= \gamma^\nu \partial_{\dot{x}^\nu} A_\mu \dot{x}^\mu \\
&= \gamma^\nu A_\mu {\delta^\mu}_\nu \\
&= \gamma^\mu A_\mu \\
&= A \\
\frac{d}{d\tau} &= \dot{x}^\mu \partial_\mu
\end{aligned}$$

Putting all this back together

$$\begin{aligned}
\nabla \mathcal{L} &= \frac{d}{d\tau}(\nabla_v \mathcal{L}) \\
\kappa \gamma^\nu \dot{x}^\mu \partial_\nu A_\mu &= \frac{d}{d\tau}\left(mv + \kappa A\right) \\
\implies & \\
\dot{p} &= \kappa \left(\gamma^\nu \dot{x}^\mu \partial_\nu A_\mu - \dot{x}^\nu \partial_\nu \gamma^\mu A_\mu\right) \\
&= \kappa \partial_\nu A_\mu \left(\gamma^\nu \dot{x}^\mu - \dot{x}^\nu \gamma^\mu\right)
\end{aligned}$$

We know this will be related to $F \cdot v$, where $F = \nabla \wedge A$. Expanding that for comparision

$$
\begin{aligned}
F \cdot v &= (\nabla \wedge A) \cdot v \\
&= (\gamma^\mu \wedge \gamma^\nu) \cdot \gamma_\alpha \dot{x}^\alpha \partial_\mu A_\nu \\
&= (\gamma^\mu {\delta^\nu}_\alpha - \gamma^\nu {\delta^\mu}_\alpha) \, \dot{x}^\alpha \partial_\mu A_\nu \\
&= \gamma^\mu \dot{x}^\nu \partial_\mu A_\nu - \gamma^\nu \dot{x}^\mu \partial_\mu A_\nu \\
&= \partial_\nu A_\mu \, (\gamma^\nu \dot{x}^\mu - \gamma^\mu \dot{x}^\nu)
\end{aligned}
$$

With the insertion of the κ factor this is an exact match, but working backwards to demonstrate that would have been harder. The equation of motion associated with the Lagrangian of equation 1 is thus

$$
\dot{p} = \kappa F \cdot v. \tag{3}
$$

3 Correspondance with classical form.

A reasonable approach to fix the constant κ is to put this into correspondance with the classical vector form of the Lorentz force equation.

Introduce a rest observer, with worldline $x = cte_0$. Computation of the spatial parts of the four vector force equation 3 for this rest observer requires taking the wedge product with the observer velocity $v = c\gamma e_0$. For clarity, for the observer frame we use a different set of basis vectors $\{e_\mu\}$, to point out that γ_0 of the derivation above does not have to equal e_0. Since the end result of the Lagrangian calculation ended up being coordinate and signature free, this is perhaps superfluous.

First calculate the field velocity product in terms of electric and magnetic components. In this new frame of reference write the proper velocity of the charged particle as $v = e_\mu \dot{f}^\mu$

$$
\begin{aligned}
F \cdot v &= (\mathbf{E} + Ic\mathbf{B}) \cdot v \\
&= (E^i e_{i0} - \epsilon_{ijk} c B^k e_{ij}) \cdot e_\mu \dot{f}^\mu \\
&= E^i \dot{f}^0 e_{i0} \cdot e_0 + E^i \dot{f}^j e_{i0} \cdot e_j - \epsilon_{ijk} c B^k \dot{f}^m e_{ij} \cdot e_m
\end{aligned}
$$

Omitting the scale factor $\gamma = dt/d\tau$ for now, application of a wedge with e_0 operation to both sides of 3 will suffice to determine this observer dependent expression of the force.

$$
\begin{aligned}
(F \cdot v) \wedge e_0 &= \left(E^i \dot{f}^0 (e_{i0} \cdot e_0) + E^i \dot{f}^j (e_{i0} \cdot e_j) - \epsilon_{ijk} c B^k \dot{f}^m e_{ij} \cdot e_m \right) \wedge e_0 \\
&= E^i \dot{f}^0 e_{i0} (e_0)^2 - \epsilon_{ijk} c B^k \dot{f}^m (e_i)^2 (e_i \delta_{jm} - e_j \delta_{im}) \wedge e_0 \\
&= (e_0)^2 \left(E^i \dot{f}^0 e_{i0} + \epsilon_{ijk} c B^k \left(\dot{f}^j e_{i0} - \dot{f}^i e_{j0} \right) \right)
\end{aligned}
$$

This wedge application has discarded the timelike components of the force equation with respect to this observer rest frame. Introduce the basis $\{\sigma_i = e_i \wedge e_0\}$ for this observers' Euclidian space. These spacetime bivectors square to unity, and thus behave in every respect like Euclidian space vector basis vectors. Writing $\mathbf{E} = E^i \sigma_i$, $\mathbf{B} = B^i \sigma_i$, and $\mathbf{v} = \sigma_i dx^i/dt$ we have

$$
(F \cdot v) \wedge e_0 = (e_0)^2 c \frac{dt}{d\tau} \left(\mathbf{E} + \epsilon_{ijk} B^k \left(\frac{df^j}{dt} \sigma_i - \frac{df^i}{dt} \sigma_j \right) \right)
$$

This inner antisymmetric sum is just the cross product. This can be observed by expanding the determinant

$$
\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \sigma_1 & \sigma_2 & \sigma_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
&= \sigma_1 (a_2 b_3 - a_3 b_2) + \sigma_2 (a_3 b_1 - a_1 b_3) + \sigma_3 (a_1 b_2 - a_2 b_1) \\
&= \sigma_i a_j b_k
\end{aligned}
$$

This leaves

$$
\kappa (F \cdot v) \wedge e_0 = \kappa (e_0)^2 c \frac{dt}{d\tau} \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right) \tag{4}
$$

Next expand the left hand side acceleration term in coordinates, and wedge with e_0

$$
\begin{aligned}
\dot{p} \wedge e_0 &= \left(e_\mu \frac{dm\dot{f}^\mu}{dt} \frac{dt}{d\tau} \right) \wedge e_0 \\
&= e_{i0} \frac{dm\dot{f}^i}{dt} \frac{dt}{d\tau}.
\end{aligned}
$$

Equating with 4, with cancelation of the $\gamma = dt/d\tau$ factors, leaves the traditional Lorentz force law in observer dependent form

$$
\frac{d}{dt} \left(m \gamma \mathbf{v} \right) = \kappa (e_0)^2 c \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right)
$$

This supplies the undetermined constant factor from the Lagrangian $\kappa(e_0)^2 c = q$. A summary statement of the results is as follows

$$\mathcal{L} = \frac{1}{2} m v^2 + q (e_0)^2 A \cdot (v/c) \tag{5}$$
$$\dot{p} = (e_0)^2 q F \cdot (v/c) \tag{6}$$
$$\mathbf{p} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{7}$$

For $(e_0)^2 = 1$, we have the proper Lorentz force equation as found in [Doran and Lasenby(2003)], which also uses the positive time signature. In that text the equation was obtained using some subtle relativistic symmetry arguments not especially easy to follow.

4 General potential.

Having written this, it would be more natural to couple the signature dependency into the velocity term of the Lagrangian since that squared velocity was the signature dependent term to start with

$$\mathcal{L} = \frac{1}{2} m v^2 (e_0)^2 + q A \cdot (v/c)$$

Although this doesn't change the equations of motion we can keep that signature factor with the velocity term. Consider a general potential as an example

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2} m v^2 (\gamma_0)^2 + \phi \\
\nabla \mathcal{L} &= \frac{d}{d\tau} (\nabla_v \mathcal{L}) \\
\frac{d}{d\tau} (m v (\gamma_0)^2) &= \nabla \phi - \frac{d}{d\tau} (\nabla_v \phi)
\end{aligned}$$

References

[Doran and Lasenby(2003)] C. Doran and A.N. Lasenby. *Geometric algebra for physicists*. Cambridge University Press New York, 2003.

[Joot(a)] Peeter Joot. Canonical momentum phrase from goldstein explored. "`http://sites.google.com/site/peeterjoot/geometric-algebra/canonical_momentum.pdf`", a.

[Joot(b)] Peeter Joot. Lagrangian derivation of lorentz force law in sta form. "http://sites.google.com/site/peeterjoot/geometric-algebra/sr_lagrangian.pdf", b.