Equations of motion given mass variation with spacetime position.

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Let

$$x = \sum \gamma_\mu x^\mu$$
$$v = \frac{dx}{d\tau} = \sum \gamma_\mu \dot{x}^\mu$$

Where whatever spacetime basis you pick has a corresponding reciprocal frame defined implicitly by:

$$\gamma^\mu \cdot \gamma_\nu = \delta^\mu{}_\nu$$

You could for example pick these so that these are othornormal with:

$$\gamma_i^2 = \gamma_i \cdot \gamma_i = -1$$
$$\gamma^i = -\gamma_i$$
$$\gamma^0 = \gamma_0$$
$$\gamma_0^2 = 1$$
$$\gamma_i \cdot \gamma_0 = 0$$

ie: the frame vectors define the metric tensor implicitly:

$$g_{\mu\nu} = \gamma_\mu \cdot \gamma_\nu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{1}$$

Now, my assumption is that given a Lagrangian of the form:

$$\mathcal{L} = \frac{1}{2} m v^2 + \phi \tag{2}$$

That the equations of motion follow by computation of:

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = \frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \tag{3}$$

I don't have any proof of this (I don't yet know any calculus of variations, and this is a guess based on intuition). It does however work out to get the covariant form of the Lorentz force law, so I think it is right.

To get the EOM we need the squared proper velocity. This is just c^2. Example: for an orthonormal spacetime frame one has:

$$\begin{aligned}
v^2 &= \left(\gamma^0 c dt/d\tau + \sum \gamma_i dx/d\tau\right)^2 \\
&= \gamma \left(\gamma_0 c + \sum \gamma_i dx/dt\right)^2 \\
&= \gamma^2 \left(c^2 - \mathbf{v}^2\right) = c^2
\end{aligned}$$

but if we leave this expressed in terms of coordinates (also don't have to assume the diagonal metric tensor, since we can use non-orthonormal basis vectors if desired) we have:

$$\begin{aligned}
v^2 &= \left(\sum \gamma_\mu \dot{x}^\mu\right) \cdot \left(\sum \gamma_\nu \dot{x}^\nu\right) \\
&= \sum \gamma_\mu \cdot \gamma_\nu \dot{x}^\mu \dot{x}^\nu \\
&= \sum g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu
\end{aligned}$$

Therefore the Lagrangian to minimize is:

$$\mathcal{L} = \frac{1}{2} m \sum g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \phi. \tag{4}$$

Performing the calculations for the EOM, and in this case, also allowing mass to be a function of space or time position ($m = m(x^\mu)$)

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x^\mu} &= \frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \\
\frac{\partial \phi}{\partial x^\mu} + \frac{1}{2}\frac{\partial m}{\partial x^\mu}\sum g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta &= \\
\frac{\partial \phi}{\partial x^\mu} + \frac{1}{2}\frac{\partial m}{\partial x^\mu} v^2 &= \\
&= \frac{1}{2}\frac{d}{d\tau} m \sum g_{\alpha\beta}\frac{\partial}{\partial x^\mu}\left(\dot{x}^\alpha \dot{x}^\beta\right) \\
&= \frac{1}{2}\frac{d}{d\tau} m \sum g_{\alpha\beta}\left(\delta^{\mu\alpha}\dot{x}^\beta + \dot{x}^\alpha \delta^{\mu\beta}\right) \\
&= \frac{d}{d\tau} m \sum g_{\alpha\mu}\dot{x}^\alpha \\
&= \sum \frac{\partial m}{\partial x^\beta}\dot{x}^\beta g_{\alpha\mu}\dot{x}^\alpha + m g_{\alpha\mu}\ddot{x}^\alpha
\end{aligned}
$$

Now, the metric tensor values can be removed by summing since they can be used to switch upper and lower indexes of the frame vectors:

$$
\begin{aligned}
\gamma_\mu &= \sum a^\nu \gamma^\nu \\
\gamma_\mu \cdot \gamma_\beta &= \sum a^\nu \gamma^\nu \cdot \gamma_\beta \\
&= \sum a^\nu \delta^\nu{}_\beta \\
&= a^\beta \\
\implies & \\
\gamma_\mu &= \sum \gamma_\mu \cdot \gamma_\nu \gamma^\nu \\
&= \sum g_{\mu\nu}\gamma^\nu
\end{aligned}
$$

If you are already familiar with tensors then this may be obvious to you (but wasn't to me with only vector background).

Multiplying throughout by γ^μ, and summing over μ one has:

$$
\begin{aligned}
\sum \gamma^\mu \left(\frac{\partial \phi}{\partial x^\mu} + \frac{1}{2}\frac{\partial m}{\partial x^\mu} v^2\right) &= \sum \gamma^\mu \left(\frac{\partial m}{\partial x^\beta}\dot{x}^\beta g_{\alpha\mu}\dot{x}^\alpha + m g_{\alpha\mu}\ddot{x}^\alpha\right) \\
+ \left(\sum \gamma^\mu \frac{\partial}{\partial x^\mu}\right)\phi + \frac{1}{2}v^2\left(\sum \gamma^\mu \frac{\partial}{\partial x^\mu}\right) m &= \\
&= \sum \frac{\partial m}{\partial x^\beta}\dot{x}^\beta \gamma^\mu \gamma_\alpha \cdot \gamma_\mu \dot{x}^\alpha + m\gamma^\mu \gamma_\alpha \cdot \gamma_\mu \ddot{x}^\alpha \\
&= \sum \frac{\partial m}{\partial x^\beta}\dot{x}^\beta \gamma_\alpha \dot{x}^\alpha + m\gamma_\alpha \ddot{x}^\alpha
\end{aligned}
$$

Writing:

$$\nabla = \sum \gamma^{\mu} \frac{\partial}{\partial x^{\mu}}$$

This is:

$$\nabla \phi + \frac{1}{2} v^2 \nabla m = v \sum \frac{\partial m}{\partial x^{\beta}} \dot{x}^{\beta} + m\dot{v}$$

However,

$$\begin{aligned}
(\nabla m) \cdot v &= \left(\sum \gamma^{\mu} \frac{\partial m}{\partial x^{\mu}} \right) \cdot \left(\sum \gamma_{\nu} \dot{x}^{\nu} \right) \\
&= \sum \gamma^{\mu} \cdot \gamma_{\nu} \frac{\partial m}{\partial x^{\mu}} \dot{x}^{\nu} \\
&= \sum \delta^{\mu}{}_{\nu} \frac{\partial m}{\partial x^{\mu}} \dot{x}^{\nu} \\
&= \sum \frac{\partial m}{\partial x^{\mu}} \dot{x}^{\mu} = \frac{dm}{d\tau}
\end{aligned}$$

That allows for expressing the EOM in strict vector form:

$$\nabla \phi + \frac{1}{2} v^2 \nabla m = v \nabla m \cdot v + m\dot{v}. \tag{5}$$

However, there is still an asymmetry here, as one would expect a $\dot{m}v$ term. Regrouping slightly, and using some algebraic vector manipulation we have:

$$
\begin{aligned}
m\dot{v} + v\nabla m \cdot v - \frac{1}{2}v^2\nabla m &= \nabla\phi \\
m\dot{v} + \frac{1}{2}v(\underbrace{2\nabla m \cdot v - v\nabla m}_{2a\cdot b - ba = ab}) &= \\
m\dot{v} + \frac{1}{2}v(\nabla m)v &= \\
m\dot{v} + \frac{1}{2}(v\nabla m)v &= \\
m\dot{v} + \frac{1}{2}(2v\cdot\nabla m - \nabla m v)v &= \\
m\dot{v} + (v\cdot\nabla m)v - \frac{1}{2}(\nabla m v)v &= \\
m\dot{v} + \dot{m}v - \frac{1}{2}\nabla m(vv) &= \\
&\implies \\
\frac{d(mv)}{d\tau} = m\dot{v} + \dot{m}v &= \frac{1}{2}\nabla mc^2 + \nabla\phi \\
&= \nabla\left(\phi - \frac{1}{2}mc^2\right) \\
&= \nabla\left(\phi - \frac{1}{2}mv^2\right)
\end{aligned}
$$

So, after a whole wack of algebra, the end result is to show the proper time varient of the Lagrangian equations imply that our proper force can be expressed as a (spacetime) gradient.

The cavaet is that if the mass is allowed to vary, it also needs to be included in the generalized potential associated with the equation of motion.

1.1 Summarizing.

We took this Lagrangian with kinetic energy and non-velocity dependent potential terms, where the mass in the kinetic energy term is allowed to vary with position or time. That plus the presumed proper-time Lagrange equations:

$$
\mathcal{L} = \frac{1}{2}mv^2 + \phi \tag{6}
$$

$$
\frac{\partial\mathcal{L}}{\partial x^\mu} = \frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}, \tag{7}
$$

when followed to their algebraic conclusion together imply that the equation of motion is:

$$\frac{d(mv)}{d\tau} = \nabla\mathcal{L}, \tag{8}$$

2 Examine spatial components for comparision with Newtonian limit.

Now, in the original version of this document, the signs for all the ϕ terms were inverted. This was changed since we want agreement with the newtonian limit, and there is an implied sign change hiding in the above equations.

Consider, the constant mass case, where the Lagrangian is specified in terms of spatial quantities:

$$\mathcal{L} = \frac{1}{2}mv^2 + \phi = \frac{1}{2}m\gamma^2(c^2 - \mathbf{v}^2) = \frac{1}{2}m\gamma^2c^2 - \gamma^2\left(\frac{1}{2}m\mathbf{v}^2 - \phi\right)$$

For $|\mathbf{v}| << c$, $\gamma \approx 1$, so we have a constant term in the Lagrangian of $\frac{1}{2}mc^2$ which won't change the EOM and can be removed. The remainder is our normal kinetic minus potential Lagrangian (the sign inversion on the entire remainding Lagrangian also won't change the EOM result).

Suppose one picks an orthonormal spacetime frame as given in the example metric tensor of equation 1. To select our spatial quantities we wedge with γ_0.

For the left hand side of our equation of motion 8 we have:

$$\begin{aligned}
\frac{d(mv)}{d\tau} \wedge \gamma_0 &= \frac{d(mv)\wedge\gamma_0}{dt}\frac{dt}{d\tau} \\
&= \frac{dp\wedge\gamma_0}{dt}\frac{dt}{d\tau} \\
&= \frac{dt}{d\tau}\frac{d}{dt}m(c\gamma_0 + \sum\gamma_i\dot{x}^i)\wedge\gamma_0 \\
&= \frac{dt}{d\tau}\frac{d}{dt}m\sum(\gamma_i\wedge\gamma_0)\dot{x}^i \\
&= \frac{dt}{d\tau}\frac{d}{dt}m\sum\sigma_i\dot{x}^i \\
&= \frac{dt}{d\tau}\frac{d}{dt}(m\mathbf{v}\gamma) \\
&= \gamma\frac{d(\gamma\mathbf{p})}{dt}
\end{aligned}$$

Now, looking at the right hand side of the EOM we have (again for the constant mass case where we expect agreement with our familiar Newtonian EOM):

$$
\begin{aligned}
\nabla \left(\phi - \frac{1}{2}mv^2\right) \wedge \gamma_0 &= (\nabla\phi) \wedge \gamma_0 \\
&= \sum \gamma^\mu \wedge \gamma_0 \frac{\partial\phi}{\partial x^\mu} \\
&= \sum \gamma^i \wedge \gamma_0 \frac{\partial\phi}{\partial x^i} \\
&= -\sum \gamma_i \wedge \gamma_0 \frac{\partial\phi}{\partial x^i} \\
&= -\sum \sigma_i \frac{\partial\phi}{\partial x^i} \\
&= -\boldsymbol{\nabla}\phi
\end{aligned}
$$

Therefore in the limit $|\mathbf{v}| << c$ we have our agreement with the Newtonian EOM:

$$
\gamma \frac{d(\gamma \mathbf{p})}{dt} = -\boldsymbol{\nabla}\phi \approx \frac{d\mathbf{p}}{dt} \tag{9}
$$