Field form of Noether's Law.

Peeter Joot peeter.joot@gmail.com

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1 Derivation.

It was seen in [Joot(b)] that Noether's law for a line integral action was shown to essentially be an application of the chain rule, coupled with an application of the Euler-Lagrange equations.

For a field Lagrangian a similar conservation statement can be made, where it takes the form of a divergence relationship instead of derivative with respect to the integration parameter associated with the line integral.

The following derivation follows [Doran and Lasenby(2003)], but is dumbed down to the scalar field variable case, and additional details are added.

The Lagrangian to be considered is

$$\mathcal{L} = \mathcal{L}(\psi, \partial_\mu \psi),$$

and the single field case is sufficent to see how this works. Consider the following transformation:

$$
\begin{aligned}
\psi &\to f(\psi, \alpha) = \psi' \\
\mathcal{L}' &= \mathcal{L}(f, \partial_\mu f).
\end{aligned}
$$

Taking derivatives of the transformed Lagrangian with respect to the free transformation variable α, we have

$$
\frac{d\mathcal{L}'}{d\alpha} = \frac{\partial \mathcal{L}}{\partial f}\frac{\partial f}{\partial \alpha} + \sum_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu f)}\frac{\partial(\partial_\mu f)}{\partial \alpha} \tag{1}
$$

The Euler-Lagrange field equations for the transformed Lagrangian are

$$
\frac{\partial \mathcal{L}}{\partial f} = \sum_\mu \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu f)}. \tag{2}
$$

For some for background discussion, examples, and derivation of the field form of Noether's equation see [Joot(c)].

Now substitute back into 1 for

$$
\begin{aligned}
\frac{d\mathcal{L}'}{d\alpha} &= \sum_\mu \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu f)}\right)\frac{\partial f}{\partial \alpha} + \sum_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu f)}\frac{\partial(\partial_\mu f)}{\partial \alpha} \\
&= \sum_\mu \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu f)}\right)\frac{\partial f}{\partial \alpha} + \sum_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu f)}\partial_\mu \frac{\partial f}{\partial \alpha}
\end{aligned}
$$

Using the product rule we have

$$
\begin{aligned}
\frac{d\mathcal{L}'}{d\alpha} &= \sum_\mu \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu f)}\frac{\partial f}{\partial \alpha}\right) \\
&= \sum_\mu \gamma^\mu \partial_\mu \cdot \left(\gamma_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu f)}\frac{\partial f}{\partial \alpha}\right) \\
&= \nabla \cdot \left(\gamma_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi')}\frac{\partial \psi'}{\partial \alpha}\right)
\end{aligned}
$$

Here the field doesn't have to be a relativistic field which could be implied by the use of the standard symbols for relativistic four vector basis $\{\gamma_\mu\}$ of STA. This is really a statement that one can form a gradient in the field variable configuration space using any appropriate reciprocal basis pair.

Noether's law for a field Lagrangian is a statement that if the transformed Lagrangian is unchanged (invariant) by some type of parameterized field variable transformation, then with $J' = J'^\mu \gamma_\mu$ one has

$$\frac{d\mathcal{L}'}{d\alpha} = \nabla \cdot J' = 0 \tag{3}$$

$$J'^{\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi')} \frac{\partial \psi'}{\partial \alpha} \tag{4}$$

FIXME: GAFP evaluates things at $\alpha = 0$ where that is the identity case. I think this is what allows them to drop the primes later. Must think this through.

2 Examples.

2.1 Schrödinger invariance under phase change.

The relativistic Schrödinger Lagrangian

$$\mathcal{L} = \eta^{\mu\nu}\partial_\mu\psi\partial_\nu\psi^* + m^2\psi\psi^*,$$

gives a simple example application of the field form of Noether's equation, for a transformation that involves a phase change

$$\begin{aligned}\psi &\rightarrow \psi' = e^{i\theta}\psi \\ \psi^* &\rightarrow \psi^{*\prime} = e^{-i\theta}\psi^*.\end{aligned}$$

This transformation leaves the Lagrangian unchanged, so there is an associated conserved quantity.

$$\begin{aligned}\frac{\partial \psi'}{\partial \theta} &= i\psi' \\ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi')} &= \eta^{\mu\nu}\partial_\nu \psi'^* = \partial^\mu \psi'^*\end{aligned}$$

Summing all the field partials, treating ψ, and ψ^* as separate field variables the divergence conservation statement is

$$\partial_\mu \left(\underbrace{\partial^\mu \psi'^* i\psi' - \partial^\mu \psi' i\psi'^*}_{J'^\mu} \right) = 0$$

Dropping primes and writing $J = \gamma_\mu J^\mu$, this is

$$\begin{aligned}J &= i(\psi\nabla\psi^* - \psi^*\nabla\psi) \\ \nabla \cdot J &= 0\end{aligned}$$

Apparently with charge added this quantity actually represents electric current density. It will be interesting to learn some quantum mechanics and see how this works.

2.2 Lorentz boost and rotation invariance of Maxwell Lagrangian.

$$\begin{aligned}
\mathcal{L} &= -\left\langle (\nabla \wedge A)^2 \right\rangle + \kappa A \cdot J && (5)\\
&= \partial_\mu A_\nu (\partial^\mu A^\nu - \partial^\nu A^\mu) + \kappa A_\sigma J^\sigma && (6)\\
\kappa &= \frac{2}{\epsilon_0 c} && (7)
\end{aligned}$$

The rotation and boost invariance of the Maxwell Lagragian was demonstrated in [Joot(a)].

Following [Joot(d)] write the Lorentz boost or rotation in exponential form.

$$L(x) = \exp(-\alpha i/2) x \exp(\alpha i/2), \quad \Lambda = \exp(-\alpha i/2)$$

where i is a unit spatial bivector for a rotation of $-\alpha$ radians, and a boost with rapidity α when i is a spacetime unit bivector.

Introducing the transformation

$$A \rightarrow A' = \Lambda A \Lambda^\dagger$$

The change in A' with respect to α is

$$\frac{\partial A'}{\partial \alpha} = -iA' + A'i = 2A' \cdot i = 2{A'}_\sigma \gamma^\sigma \cdot i$$

Next we want to compute

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial(\partial_\mu {A'}_\nu)} &= \frac{\partial}{\partial(\partial_\mu {A'}_\nu)} \left(\partial_\alpha {A'}_\beta(\partial^\alpha {A'}^\beta - \partial^\beta {A'}^\alpha) + \kappa {A'}_\sigma J^\sigma\right) \\
&= \left(\frac{\partial}{\partial(\partial_\mu {A'}_\nu)} \partial_\alpha {A'}_\beta\right) \left(\partial^\alpha {A'}^\beta - \partial^\beta {A'}^\alpha\right) \\
&+ \partial^\alpha {A'}^\beta \frac{\partial}{\partial(\partial_\mu {A'}_\nu)} \left(\partial_\alpha {A'}_\beta - \partial_\beta {A'}_\alpha)\right) \\
&= \left(\frac{\partial}{\partial(\partial_\mu {A'}_\nu)} \partial_\mu {A'}_\nu\right) \left(\partial^\mu {A'}^\nu - \partial^\nu {A'}^\mu\right) \\
&+ \partial^\mu {A'}^\nu \frac{\partial}{\partial(\partial_\mu {A'}_\nu)} \partial_\mu {A'}_\nu \\
&- \partial^\nu {A'}^\mu \frac{\partial}{\partial(\partial_\mu {A'}_\nu)} \partial_\mu {A'}_\nu \\
&= 2\left(\partial^\mu {A'}^\nu - \partial^\nu {A'}^\mu\right) \\
&= 2F^{\mu\nu}
\end{aligned}$$

Employing the vector field form of Noether's equation as in 16 the conserved current C components are

$$\begin{aligned}
C^\mu &= 2(\gamma_\nu F^{\mu\nu}) \cdot (2A \cdot i) \\
&\propto (\gamma_\nu F^{\mu\nu}) \cdot (A \cdot i) \\
&\propto (\gamma^\mu \cdot F) \cdot (A \cdot i)
\end{aligned}$$

Or

$$C = \gamma_\mu((\gamma^\mu \cdot F) \cdot (A \cdot i)) \tag{8}$$

Here C was used instead of J for the conserved current vector since J is already taken for the current charge density itself.

2.3 Questions.

FIXME: What is this quantity? It has the look of angular momentum, or torque, or an inertial tensor. Does it have a physical significance? Can the i be factored out of the expression, leaving a conserved quantity that is some linear function only of F, and A (this was possible in the Lorentz force Lagrangian for the same invariance considerations).

2.4 Expansion for x-axis boost.

As an example to get a feel for 8, lets expand this for a specific spacetime boost plane. Using the x-axis that is $i = \gamma_1 \wedge \gamma_0$

First expanding the potential projection one has

$$\begin{aligned}A \cdot i &= (A_\mu \gamma^\mu) \cdot (\gamma_1 \wedge \gamma_0) \\ &= A_1 \gamma_0 - A_0 \gamma_1.\end{aligned}$$

Next the μ component of the field is

$$\begin{aligned}\gamma^\mu \cdot F &= \frac{1}{2} F^{\alpha\beta} \gamma^\mu \cdot (\gamma_\alpha \wedge \gamma_\beta) \\ &= \frac{1}{2} F^{\mu\beta} \gamma_\beta - \frac{1}{2} F^{\alpha\mu} \gamma_\alpha \\ &= F^{\mu\alpha} \gamma_\alpha\end{aligned}$$

So the μ component of the conserved vector is

$$\begin{aligned}C^\mu &= (\gamma^\mu \cdot F) \cdot (A \cdot i) \\ &= (F^{\mu\alpha} \gamma_\alpha) \cdot (A_1 \gamma_0 - A_0 \gamma_1) \\ &= (F^{\mu\alpha} \gamma_\alpha) \cdot (A^0 \gamma^1 - A^1 \gamma^0)\end{aligned}$$

Therefore the conservation statement is

$$C^\mu = F^{\mu 1} A^0 - F^{\mu 0} A^1 \tag{9}$$

$$\partial_\mu C^\mu = 0 \tag{10}$$

Let's write out the components of 9 explicitly, to perhaps get a better feel for them.

$$\begin{aligned}C^0 &= F^{01} A^0 = -E_x \phi \\ C^1 &= -F^{10} A^1 = -E_x A_x \\ C^2 &= F^{21} A^0 - F^{20} A^1 = B_z \phi - E_y A_x \\ C^3 &= F^{31} A^0 - F^{30} A^1 = -B_y \phi - E_z A_x\end{aligned}$$

Well, that's not particularily enlightening looking after all.

2.5 Expansion for rotation or boost.

Suppose that one takes $i = \gamma^\mu \wedge \gamma^\nu$, so that we have a symmetry for a boost if one of μ or ν is zero, and rotational symmetry otherwise.

This gives

$$
\begin{aligned}
A \cdot i &= (A^\alpha \gamma_\alpha) \cdot (\gamma^\mu \wedge \gamma^\nu) \\
&= A^\mu \gamma^\nu - A^\nu \gamma^\mu
\end{aligned}
$$

$$
\begin{aligned}
C^\alpha &= (\gamma^\alpha \cdot F) \cdot (A \cdot i) \\
&= (F^{\alpha\beta} \gamma_\beta) \cdot (A^\mu \gamma^\nu - A^\nu \gamma^\mu)
\end{aligned}
$$

$$
C^\alpha = F^{\alpha\nu} A^\mu - F^{\alpha\mu} A^\nu \tag{11}
$$

For a rotation in the a, b, plane with $\mu = a$, and $\nu = b$ (say), lets write out the C^α components explicitly in terms of **E** and **B** components, also writing $0 < d$, $a \neq d \neq b$. That is

$$
\begin{aligned}
C^0 &= F^{0b} A^a - F^{0a} A^b = E^a A^b - E^b A^a \\
C^1 &= F^{1b} A^a - F^{1a} A^b \\
C^2 &= F^{2b} A^a - F^{2a} A^b \\
C^3 &= F^{3b} A^a - F^{3a} A^b
\end{aligned}
$$

Only the first term of this reduces nicely. Suppose we additionally write $a = 1$, $b = 2$ to make things more concrete. Then we have

$$
\begin{aligned}
C^0 &= F^{02} A^1 - F^{01} A^2 = E_x A_y - E_y A_x = (\mathbf{E} \times \mathbf{A})_z \\
C^1 &= F^{12} A^1 - F^{11} A^2 = -B_z A_x \\
C^2 &= F^{22} A^1 - F^{21} A^2 = B_z A_x \\
C^3 &= F^{32} A^1 - F^{31} A^2 = B_x A_x + B_y A_y
\end{aligned}
$$

The timelike component of whatever this vector is the z component of a cross product (spatial component of the $\mathbf{E} \times \mathbf{A}$ product in the direction of the normal to the rotational plane), but what's the rest?

2.5.1 Conservation statement.

Returning to 11, the conservation statement can be calculated as

$$
\begin{aligned}
0 &= \partial_\alpha C^\alpha \\
&= \partial_\alpha F^{\alpha\nu} A^\mu - \partial_\alpha F^{\alpha\mu} A^\nu + F^{\alpha\nu} \partial_\alpha A^\mu - F^{\alpha\mu} \partial_\alpha A^\nu
\end{aligned}
$$

But the grade one terms of the Maxwell equation in tensor form is

$$
\partial_\mu F^{\mu\alpha} = J^\alpha / \epsilon_0 c
$$

So we have

$$
\begin{aligned}
0 &= \frac{1}{\epsilon_0 c} \left(J^\nu A^\mu - J^\mu A^\nu \right) + {F_\alpha}^\nu \partial^\alpha A^\mu - {F_\alpha}^\mu \partial^\alpha A^\nu \\
&= \frac{1}{\epsilon_0 c} \left(J^\nu A^\mu - J^\mu A^\nu \right) + {F_\alpha}^\nu F^{\alpha\mu} - {F_\alpha}^\mu F^{\alpha\nu}
\end{aligned}
$$

This first part is some sort of current-potential torque like beastie. That second part, the squared field term is what? I don't see an obvious way to reduce it to something more structured.

3 Appendix.

3.1 Multivariable derivation.

For completion sake, cut and pasted with with most discussion omitted, the multiple field variable case follows in the same fashion as the single field variable Lagrangian.

$$
\mathcal{L} = \mathcal{L}(\psi_\sigma, \partial_\mu \psi_\sigma),
$$

The transformation is now:

$$
\begin{aligned}
\psi_\sigma &\rightarrow f_\sigma(\psi_\sigma, \alpha) = \psi'_\sigma \\
\mathcal{L}' &= \mathcal{L}(f_\sigma, \partial_\mu f_\sigma).
\end{aligned}
$$

Taking derivatives:

$$
\frac{d\mathcal{L}'}{d\alpha} = \sum_\sigma \frac{\partial \mathcal{L}}{\partial f_\sigma} \frac{\partial f_\sigma}{\partial \alpha} + \sum_{\mu,\sigma} \frac{\partial \mathcal{L}}{\partial (\partial_\mu f_\sigma)} \frac{\partial (\partial_\mu f_\sigma)}{\partial \alpha} \tag{12}
$$

Again, making the Euler-Lagrange substitution of 2 (with $f \rightarrow f_\sigma$) back into 12 gives

$$\begin{aligned}
\frac{d\mathcal{L}'}{d\alpha} &= \sum_\sigma \left(\sum_\mu \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu f_\sigma)} \right) \frac{\partial f_\sigma}{\partial \alpha} + \sum_{\mu,\sigma} \frac{\partial \mathcal{L}}{\partial(\partial_\mu f_\sigma)} \frac{\partial(\partial_\mu f_\sigma)}{\partial \alpha} \\
&= \sum_{\mu,\sigma} \left(\left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu f_\sigma)} \right) \frac{\partial f_\sigma}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu f_\sigma)} \partial_\mu \frac{\partial f_\sigma}{\partial \alpha} \right) \\
&= \sum_{\mu,\sigma} \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu f_\sigma)} \frac{\partial f_\sigma}{\partial \alpha} \right) \\
&= \sum_\mu \gamma^\mu \partial_\mu \cdot \left(\sum_{\sigma,\nu} \gamma_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu f_\sigma)} \frac{\partial f_\sigma}{\partial \alpha} \right) \\
&= \nabla \cdot \left(\sum_{\sigma,\nu} \gamma_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu \psi'_\sigma)} \frac{\partial \psi'_\sigma}{\partial \alpha} \right)
\end{aligned}$$

Or

$$\frac{d\mathcal{L}'}{d\alpha} = \nabla \cdot J' = 0 \tag{13}$$

$$J' = J'^\mu \gamma_\mu \tag{14}$$

$$J'^\mu = \sum_\sigma \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi'_\sigma)} \frac{\partial \psi'_\sigma}{\partial \alpha} \tag{15}$$

A notational convienence for vector valued fields, in particular as we have in the electrodynamic Lagrangian for the vector potential, the chain rule summation in 13 above can be replaced with a dot product.

$$J'^\mu = \gamma_\sigma \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi'_\sigma)} \cdot \frac{\partial \gamma^\sigma \psi'_\sigma}{\partial \alpha}$$

Dropping primes for convience, and writing $\Psi = \gamma^\sigma \psi_\sigma$ for the vector field variable, the field form of Noether's law takes the form

$$J = \gamma_\mu \left(\gamma_\sigma \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_\sigma)} \cdot \frac{\partial \Psi}{\partial \alpha} \right) \tag{16}$$

$$\nabla \cdot J = 0. \tag{17}$$

That is, a current vector with respect to this configuration space divergence is conserved when the Lagrangian field transformation is invariant.

References

[Doran and Lasenby(2003)] C. Doran and A.N. Lasenby. *Geometric algebra for physicists*. Cambridge University Press New York, 2003.

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