Outerpmorphism Question

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1

Geometric Algebra for Physicists example of a linear operator.

$$F(a) = a + \alpha(a \cdot f_1) f_2. \tag{1}$$

This is used to compute the determinant without putting the operator in matrix form.

1.1 bivector outermorphism.

Their first step is to compute the wedge of this function applied to two vectors. Doing this myself (not ommitting steps), I get:

$$\begin{aligned}
F(a \wedge b) &= F(a) \wedge F(b) \\
&= (a + \alpha(a \cdot f_1) f_2) \wedge (b + \alpha(b \cdot f_1) f_2) \\
&= a \wedge b + \alpha(a \cdot f_1) f_2 \wedge b + \alpha(b \cdot f_1) a \wedge f_2 + \alpha^2 (a \cdot f_1)(b \cdot f_1) \underbrace{f_2 \wedge f_2}_{=0} \\
&= a \wedge b + \alpha\left((b \cdot f_1) a - (a \cdot f_1) b\right) \wedge f_2 \\
&= a \wedge b + \alpha\left((a \wedge b) \cdot f_1\right) \wedge f_2
\end{aligned}$$

This has a very similar form to the original function F. In particular one can write

$$\begin{aligned}
F(a) &= a + \alpha(a \cdot f_1) f_2 \\
&= a + \langle \alpha(a \cdot f_1) f_2 \rangle_1 \\
&= a + \langle \alpha(a \cdot f_1) f_2 \rangle_{0+1} \\
&= a + \alpha(a \cdot f_1) \wedge f_2
\end{aligned}$$

Here the fundamental definition of the wedge product as the highest grade part of a product of blades has been used to show that the new bivector function defined via outermorphism has the same form as the original, once we put the original in the new form that applies to bivector and vector:

$$F(A) = A + \alpha(A \cdot f_1) \wedge f_2 \tag{2}$$

1.2 Induction.

Now, proceding inductively, assuming that this is true for some grade k blade A, one can calculate $F(A) \wedge F(b)$ for a vector b:

$$\begin{aligned}
F(A) \wedge F(b) &= (A + \alpha(A \cdot f_1) \wedge f_2) \wedge (b + \alpha(b \cdot f_1) f_2) \\
&= A \wedge b + \alpha(b \cdot f_1) A \wedge f_2 + \alpha((A \cdot f_1) \wedge f_2) \wedge b + \alpha^2 (b \cdot f_1)((A \cdot f_1) \wedge f_2) \wedge f_2 \\
&= A \wedge b + \alpha\left((b \cdot f_1) A - (A \cdot f_1) \wedge b\right) \wedge f_2 \\
&= A \wedge b + \alpha \langle (b \cdot f_1) A - (A \cdot f_1) b \rangle_k \wedge f_2
\end{aligned}$$

Now, similar to the bivector case, this inner quantity can be reduced, but it is messier to do so:

$$\langle (b \cdot f_1) A - (A \cdot f_1) b \rangle_k = \frac{1}{2} \left\langle b f_1 A - A f_1 b + f_1 (bA + (-1)^k A b) \right\rangle_k$$

$$\implies \langle (b \cdot f_1) A - (A \cdot f_1) b \rangle_k = \frac{1}{2} \langle b f_1 A - A f_1 b \rangle_k + \langle f_1 (b \wedge A) \rangle_k \tag{3}$$

Consider first the right hand expression:

$$\begin{aligned}
\langle f_1 (b \wedge A) \rangle_k &= f_1 \cdot (b \wedge A) \\
&= (-1)^k f_1 \cdot (A \wedge b) \\
&= (-1)^k (-1)^k (A \wedge b) \cdot f_1 \\
&= (A \wedge b) \cdot f_1
\end{aligned}$$

The right hand expression in equation 3 can be shown to equal zero. That's messier still and the calculation can be found at the end.

Using that equals zero result we now have:

$$F(A) \wedge F(b) = A \wedge b + \alpha((A \wedge b) \cdot f_1) \wedge f_2$$

This completes the induction.

1.3 Can the induction be avoided?

Now, GAFP didn't do this induction, nor even claim it was required. The statement is "It follows that", after only calculating the bivector case. Is there a reason that they would be able to make such a statement without proof that is obvious to them perhaps but not to me?

2 Appendix. Messy reduction for induction.

Q: Is there an easier way to do this?

Here we want to show that

$$\frac{1}{2}\langle bf_1A - Af_1b\rangle_k = 0$$

Expanding the innards of this expression to group A and b parts together:

$$\begin{aligned}
bf_1A - Af_1b &= (f_1b - 2b \wedge f_1)A - A(bf_1 - 2f_1 \wedge b) \\
&= f_1bA - Abf_1 - 2(b \wedge f_1)A + 2A(f_1 \wedge b) \\
&= f_1(b \cdot A + b \wedge A) - (A \cdot b + A \wedge b)f_1 \\
&- 2\left((b \wedge f_1) \cdot A + \langle (b \wedge f_1)A\rangle_k + (b \wedge f_1) \wedge A\right) \\
&+ 2\left(A \cdot (f_1 \wedge b) + \langle A(f_1 \wedge b)\rangle_k + A \wedge (f_1 \wedge b)\right)
\end{aligned}$$

the grade $k-2$, and grade $k+2$ terms of the bivector product cancel (we are also only interested in the grade-k parts so can discard them). This leaves:

$$f_1 \wedge (b \cdot A) - (A \cdot b) \wedge f_1 + f_1 \cdot (b \wedge A) - (A \wedge b) \cdot f_1 - 2\langle (b \wedge f_1)A\rangle_k + 2\langle A(f_1 \wedge b)\rangle_k$$

The bivector, blade product part of this is the antisymmetric part of that product so those two last terms can be expressed with the commutator relationship for a bivector with blade: $\langle B_2A\rangle_k = \frac{1}{2}(B_2A - AB_2)$:

$$\begin{aligned}
2\langle A(f_1 \wedge b)\rangle_k - 2\langle (b \wedge f_1)A\rangle_k &= A(f_1 \wedge b) - (f_1 \wedge b)A - (b \wedge f_1)A + A(b \wedge f_1) \\
&= A(f_1 \wedge b) - (f_1 \wedge b)A + (f_1 \wedge b)A - A(f_1 \wedge b) \\
&= 0
\end{aligned}$$

So, we now have to show that we have zero for the remainder:

$$\begin{aligned}
2\langle bf_1A - Af_1b\rangle_k &= f_1 \wedge (b \cdot A) - (A \cdot b) \wedge f_1 \\
&\quad + f_1 \cdot (b \wedge A) - (A \wedge b) \cdot f_1 \\
&= (-1)^{k-1} f_1 \wedge (A \cdot b) - (-1)^{k-1} f_1 \wedge (A \cdot b) \\
&\quad + (-1)^k f_1 \cdot (A \wedge b) - (-1)^k f_1 \cdot (A \wedge b) \\
&= 0
\end{aligned}$$

3 New observation.

Looking again, I think I see one thing that I missed. The text said they were constructing the action on a general multivector. So, perhaps they meant b to be a blade. This is a typesetting subtlety if that's the case. Let's assume that is what they meant, and that b is a grade k blade. This makes the coefficient of the scalar α in equation 4.147 :

$$a \cdot f_1 f_2 \wedge b + b \cdot f_1 a \wedge f_2 = \Big((b \cdot f_1)a + (-1)^k (a \cdot f_1) b\Big) \wedge f_2$$

whereas they have:

$$((b \cdot f_1)a - (a \cdot f_1)b) \wedge f_2$$

So, no, I think they must have intended b to be a vector, not an arbitrary grade blade.

Now, indirectly, it has been proven here that for a vectors x, y, and a grade-k blade B:

$$(A \wedge x) \cdot y = A(x \cdot y) - (A \cdot y) \wedge x \tag{4}$$

Or,

$$(A \wedge x) \cdot y = (y \cdot x)A + (-1)^k (y \cdot A) \wedge x \tag{5}$$

(changed variable names to disassociate this from the specifics of this particular example), which is a generalization of the wedge product with dot product distribution identity for vectors:

$$(a \wedge b) \cdot c = a(b \cdot c) - (a \cdot c) \wedge b \tag{6}$$

I believe I've seen a still more general form of equation 4 in a Hestenes paper, but didn't think about using it a-priori. Regardless, it doesn't really appear the the GAFP text was treating b as anything but a vector, since there would have to be a $(-1)^k$ factor on equation 4.147 for it to be general.