Radial components of vector derivitives

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1 first derivative of a radially expressed vector.

Having calculated the derivative of a unit vector, the total derivative of a radially expressed vector can be calculated

$$
\begin{aligned}
(r\hat{\mathbf{r}})' &= r'\hat{\mathbf{r}} + r\hat{\mathbf{r}}' \\
&= r'\hat{\mathbf{r}} + \hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}')
\end{aligned}
$$

There are two components. One is in the $\hat{\mathbf{r}}$ direction (linear component) and the other perpendicular to that (a rotational component) in the direction of the rejection of $\hat{\mathbf{r}}$ from $\mathbf{r}'$.

2 Second derivative of a vector

Taking second derivatives of a radially expressed vector, we have

$$
\begin{aligned}
(r\hat{\mathbf{r}})'' &= (r'\hat{\mathbf{r}} + r\hat{\mathbf{r}}')' \\
&= r''\hat{\mathbf{r}} + r'\hat{\mathbf{r}}' + (r\hat{\mathbf{r}}')' \\
&= r''\hat{\mathbf{r}} + (r'/r)\hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}') + (r\hat{\mathbf{r}}')'
\end{aligned}
$$

Expanding the last term takes a bit more work

$$
\begin{aligned}
(r\hat{\mathbf{r}}')' &= (\hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}'))' \\
&= \hat{\mathbf{r}}'(\hat{\mathbf{r}} \wedge \mathbf{r}') + \hat{\mathbf{r}}(\hat{\mathbf{r}}' \wedge \mathbf{r}') + \hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}'') \\
&= (1/r)(\hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}'))(\hat{\mathbf{r}} \wedge \mathbf{r}') + \hat{\mathbf{r}}(\hat{\mathbf{r}}' \wedge \mathbf{r}') + \hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}'') \\
&= (1/r)\hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}')^2 + \hat{\mathbf{r}}(\hat{\mathbf{r}}' \wedge \mathbf{r}') + \hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}'')
\end{aligned}
$$

There are three terms to this. One a scalar (negative) multiple of $\hat{\mathbf{r}}$, and another, the rejection of $\hat{\mathbf{r}}$ from $\mathbf{r}''$. The middle term here remains to be expanded. In particular,

$$\begin{aligned}
\hat{\mathbf{r}}' \wedge \mathbf{r}' &= \hat{\mathbf{r}}' \wedge (r\hat{\mathbf{r}}' + r'\hat{\mathbf{r}}) \\
&= r'\hat{\mathbf{r}}' \wedge \hat{\mathbf{r}} \\
&= r'/2(\hat{\mathbf{r}}'\hat{\mathbf{r}} - \hat{\mathbf{r}}\hat{\mathbf{r}}') \\
&= r'/2r((\mathbf{r}' \wedge \hat{\mathbf{r}})\hat{\mathbf{r}}\hat{\mathbf{r}} - \hat{\mathbf{r}}\hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}')) \\
&= r'/2r(\mathbf{r}' \wedge \hat{\mathbf{r}} - \hat{\mathbf{r}} \wedge \mathbf{r}') \\
&= -(r'/r)\hat{\mathbf{r}} \wedge \mathbf{r}'
\end{aligned}$$

$$\implies (r\hat{\mathbf{r}}')' = (1/r)\hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}')^2 - (r'/r)\hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}') + \hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}'')$$

$$\begin{aligned}
\implies (r\hat{\mathbf{r}})'' &= r''\hat{\mathbf{r}} + (r'/r)\hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}') + (1/r)\hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}')^2 - (r'/r)\hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}') + \hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}'') \\
&= r''\hat{\mathbf{r}} + (1/r)\hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}')^2 + \hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}'') \\
&= \hat{\mathbf{r}}\left(r'' + (1/r)(\hat{\mathbf{r}} \wedge \mathbf{r}')^2\right) + \hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}'')
\end{aligned}$$

There are two terms here that are in the $\hat{\mathbf{r}}$ direction (the bivector square is a negative scalar), and one rejective term in the direction of the component perpendicular to $\hat{\mathbf{r}}$ relative to $\mathbf{r}''$.