Reciprocal Frame Vectors

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1 Approach without Geometric Algebra.

Without employing geometric algebra, one can use the projection operation expressed as a dot product and calculate the a vector orthogonal to a set of other vectors, in the direction of a reference vector.

Such a calculation also yields $\mathbb{R}^N$ results in terms of determinants, and as a side effect produces equations for parallelogram area, parallelopiped volume and higher dimensional analogues as a side effect (without having to employ change of basis diagonalization arguments that don't work well for higher dimensional subspaces).

1.1 Orthogonal to one vector

The simplest case is the vector perpendicular to another. In anything but $\mathbb{R}^2$ there are a whole set of such vectors, so to express this as a non-set result a reference vector is required.

Calculation of the coordinate vector for this case follows directly from the dot product. Borrowing the GA term, we subtract the projection to calculate the rejection.

$$
\begin{aligned}
\mathrm{Rej}_{\hat{\mathbf{u}}}(\mathbf{v}) &= \mathbf{v} - \mathbf{v} \cdot \hat{\mathbf{u}}\hat{\mathbf{u}} \\
&= \frac{1}{\mathbf{u}^2}(\mathbf{v}\mathbf{u}^2 - \mathbf{v} \cdot \mathbf{u}\mathbf{u}) \\
&= \frac{1}{\mathbf{u}^2}\sum v_i \mathbf{e}_i u_j u_j - v_j u_j u_i \mathbf{e}_i \\
&= \frac{1}{\mathbf{u}^2}\sum u_j \mathbf{e}_i \begin{vmatrix} v_i & v_j \\ u_i & u_j \end{vmatrix} \\
&= \frac{1}{\mathbf{u}^2}\sum_{i<j} (u_i \mathbf{e}_j - u_j \mathbf{e}_i) \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix}
\end{aligned}
$$

Thus we can write the rejection of $\mathbf{v}$ from $\hat{\mathbf{u}}$ as:

$$\mathrm{Rej}_{\hat{\mathbf{u}}}\left(\mathbf{v}\right) = \frac{1}{\mathbf{u}^2} \sum_{i<j} \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix} \begin{vmatrix} u_i & u_j \\ \mathbf{e}_i & \mathbf{e}_j \end{vmatrix} \tag{1}$$

Or introducing some shorthand:

$$D_{ij}^{\mathbf{uv}} = \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix}$$
$$D_{ij}^{\mathbf{ue}} = \begin{vmatrix} u_i & u_j \\ \mathbf{e}_i & \mathbf{e}_j \end{vmatrix}$$

equation 1 can be expressed in a form that will be slightly more convient for larger sets of vectors:

$$\mathrm{Rej}_{\hat{\mathbf{u}}}\left(\mathbf{v}\right) = \frac{1}{\mathbf{u}^2} \sum_{i<j} D_{ij}^{\mathbf{uv}} D_{ij}^{\mathbf{ue}} \tag{2}$$

Note that although the GA axiom $\mathbf{u}^2 = \mathbf{u} \cdot \mathbf{u}$ has been used in equations 1 and 2 above and the derivation, that was not necessary to prove them. This can, for now, be thought of as a notational convenience, to avoid having to write $\mathbf{u} \cdot \mathbf{u}$, or $\|\mathbf{u}\|^2$.

This result can be used to express the $\mathbb{R}^N$ area of a parallelogram since we just have to multiply the length of $\mathrm{Rej}_{\hat{\mathbf{u}}}\left(\mathbf{v}\right)$:

$$\|\mathrm{Rej}_{\hat{\mathbf{u}}}\left(\mathbf{v}\right)\|^2 = \mathrm{Rej}_{\hat{\mathbf{u}}}\left(\mathbf{v}\right) \cdot \mathbf{v} = \frac{1}{\mathbf{u}^2} \sum_{i<j} \left(D_{ij}^{\mathbf{uv}}\right)^2$$

with the length of the base $\|\mathbf{u}\|$. [FIXME: insert figure.]

Thus the area (squared) is:

$$\mathrm{A}_{\mathbf{u},\mathbf{v}}^2 = \sum_{i<j} \left(D_{ij}^{\mathbf{uv}}\right)^2 \tag{3}$$

For the special case of a vector in $\mathbb{R}^2$ this is

$$\mathrm{A}_{\mathbf{u},\mathbf{v}} = |D_{12}^{\mathbf{uv}}| = \mathrm{abs}\left(\begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix}\right) \tag{4}$$

1.2 Vector orthogonal to two vectors in direction of a third.

The same procedure can be followed for three vectors, but the algebra gets messier. Given three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ we can calculate the component $\mathbf{w}'$ of $\mathbf{w}$ perpendicular to $\mathbf{u}$ and $\mathbf{v}$. That is:

$$
\begin{aligned}
\mathbf{v}' &= \mathbf{v} - \mathbf{v}\cdot\hat{\mathbf{u}}\hat{\mathbf{u}} \\
\Longrightarrow & \\
\mathbf{w}' &= \mathbf{w} - \mathbf{w}\cdot\hat{\mathbf{u}}\hat{\mathbf{u}} - \mathbf{w}\cdot\hat{\mathbf{v}}'\hat{\mathbf{v}}'
\end{aligned}
$$

After expanding this out, a number of the terms magically cancel out and one is left with

$$
\begin{aligned}
\mathbf{w}'' = \mathbf{w}'(\mathbf{u}^2\mathbf{v}^2 - (\mathbf{u}\cdot\mathbf{v})^2) = \mathbf{u}&\left(-\mathbf{u}\cdot\mathbf{w}\mathbf{v}^2 + (\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w})\right) \\
+ \mathbf{v}&\left(-\mathbf{u}^2(\mathbf{v}\cdot\mathbf{w}) - (\mathbf{u}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{w})\right) \\
+ \mathbf{w}&\left(\mathbf{u}^2\mathbf{v}^2 - (\mathbf{u}\cdot\mathbf{v})^2\right)
\end{aligned}
$$

And this in turn can be expanded in terms of coordinates and the results collected yielding

$$
\begin{aligned}
\mathbf{w}'' &= \sum \mathbf{e}_i u_j v_k \left(u_i \begin{vmatrix} v_j & v_k \\ w_j & w_k \end{vmatrix} - v_i \begin{vmatrix} u_j & u_k \\ w_j & w_k \end{vmatrix} w_i \begin{vmatrix} u_j & u_k \\ v_j & v_k \end{vmatrix} \right) \\
&= \sum \mathbf{e}_i u_j v_k \begin{vmatrix} u_i & u_j & u_k \\ v_i & v_j & v_k \\ w_i & w_j & w_k \end{vmatrix} \\
&= \sum_{i,j<k} \mathbf{e}_i \begin{vmatrix} u_j & u_k \\ v_j & v_k \end{vmatrix} \begin{vmatrix} u_i & u_j & u_k \\ v_i & v_j & v_k \\ w_i & w_j & w_k \end{vmatrix} \\
&= \left(\sum_{i<j<k} + \sum_{j<i<k} + \sum_{j<k<i} \right) \mathbf{e}_i \begin{vmatrix} u_j & u_k \\ v_j & v_k \end{vmatrix} \begin{vmatrix} u_i & u_j & u_k \\ v_i & v_j & v_k \\ w_i & w_j & w_k \end{vmatrix}.
\end{aligned}
$$

Expanding the sum of the denominator in terms of coordinates:

$$
\mathbf{u}^2\mathbf{v}^2 - (\mathbf{u}\cdot\mathbf{v})^2 = \sum_{i<j} \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix}^2
$$

and using a change of summation indexes, our final result for the vector perpendicular to two others in the direction of a third is:

$$
\mathrm{Rej}_{\hat{\mathbf{u}},\hat{\mathbf{v}}}(\mathbf{w}) = \frac{\sum_{i<j<k} \begin{vmatrix} u_i & u_j & u_k \\ v_i & v_j & v_k \\ w_i & w_j & w_k \end{vmatrix} \begin{vmatrix} u_i & u_j & u_k \\ v_i & v_j & v_k \\ \mathbf{e}_i & \mathbf{e}_j & \mathbf{e}_k \end{vmatrix}}{\sum_{i<j} \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix}^2} \tag{5}
$$

As a small aside, it is notable here to observe that span $\left\{ \begin{vmatrix} u_i & u_j \\ \mathbf{e}_i & \mathbf{e}_j \end{vmatrix} \right\}$ is the null space for the vector $\mathbf{u}$, and the set span $\left\{ \begin{vmatrix} u_i & u_j & u_k \\ v_i & v_j & v_k \\ \mathbf{e}_i & \mathbf{e}_j & \mathbf{e}_k \end{vmatrix} \right\}$ is the null space for the two vectors $\mathbf{u}$ and $\mathbf{v}$ respectively.

Since the rejection is a normal to the set of vectors it must necessarily include these cross product like determinant terms.

As in equation 2, use of a $D^{\mathbf{uvw}}_{ijk}$ notation allows for a more compact result:

$$\mathrm{Rej}_{\hat{\mathbf{u}}\hat{\mathbf{v}}}(\mathbf{w}) = \left(\sum_{i<j} \left(D^{\mathbf{uv}}_{ij} \right)^2 \right)^{-1} \sum_{i<j<k} D^{\mathbf{uvw}}_{ijk} D^{\mathbf{uve}}_{ijk} \tag{6}$$

And, as before this yields the Volume of the parallelopiped by multiplying perpendicular height:

$$\|\mathrm{Rej}_{\hat{\mathbf{u}}\hat{\mathbf{v}}}(\mathbf{w})\| = \mathrm{Rej}_{\hat{\mathbf{u}}\hat{\mathbf{v}}}(\mathbf{w}) \cdot \mathbf{w} = \left(\sum_{i<j} \left(D^{\mathbf{uv}}_{ij} \right)^2 \right)^{-1} \sum_{i<j<k} \left(D^{\mathbf{uvw}}_{ijk} \right)^2$$

by the base area.

Thus the squared volume of a parallelopiped spanned by the three vectors is:

$$\mathrm{V}^2_{\mathbf{u},\mathbf{v},\mathbf{w}} = \sum_{i<j<k} \left(D^{\mathbf{uvw}}_{ijk} \right)^2. \tag{7}$$

The simplest case is for $\mathbb{R}^3$ where we have only one summand:

$$\mathrm{V}_{\mathbf{u},\mathbf{v},\mathbf{w}} = |D^{\mathbf{uvw}}_{ijk}| = \mathrm{abs} \left(\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \right). \tag{8}$$

1.3 Generalization. Inductive Hypothesis.

There are two things to prove

1. hypervolume of parallelopiped spanned by vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$

$$\mathrm{V}^2_{\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_k} = \sum_{i_1<i_2<\cdots<i_k} \left(D^{\mathbf{u}_{i_1}\mathbf{u}_{i_2}\cdots\mathbf{u}_{i_k}}_{i_1 i_2 \cdots i_k} \right)^2 \tag{9}$$

2. Orthogonal rejection of a set of vectors in direction of another.

$$\mathrm{Rej}_{\hat{\mathbf{u}}_1\cdots\hat{\mathbf{u}}_{k-1}}(\mathbf{u}_k) = \frac{\sum_{i_1<\cdots<i_k} D^{\mathbf{u}_{i_1}\cdots\mathbf{u}_{i_k}}_{i_1\cdots i_k} D^{\mathbf{u}_{i_1}\cdots\mathbf{u}_{i_{k-1}}\mathbf{e}}_{i_1\cdots i_k}}{\sum_{i_1<\cdots<i_{k-1}} \left(D^{\mathbf{u}_{i_1}\cdots\mathbf{u}_{i_{k-1}}}_{i_1\cdots i_{k-1}} \right)^2} \tag{10}$$

I cannot recall if I ever did the inductive proof for this. Proving for the initial case is done (since it's proved for both the two and three vector cases). For the limiting case where $k = n$ it can be observed that this is normal to all the others, so the only thing to prove for that case is if the scaling provided by hypervolume equation 9 is correct.

1.4 Scaling required for reciprocal frame vector.

Presuming an inductive proof of the general result of 10 is possible, this rejection has the property

$$\mathrm{Rej}_{\hat{\mathbf{u}}_1 \cdots \hat{\mathbf{u}}_{k-1}} (\mathbf{u}_k) \cdot \mathbf{u}_i \propto \delta_{ki}$$

With the scaling factor picked so that this equals δ_{ki}, the resulting "reciprocal frame vector" is

$$\mathbf{u}^k = \frac{\sum_{i_1 < \cdots < i_k} D^{\mathbf{u}_{i_1} \cdots \mathbf{u}_{i_k}}_{i_1 \cdots i_k} D^{\mathbf{u}_{i_1} \cdots \mathbf{u}_{i_{k-1}} \mathbf{e}}_{i_1 \cdots i_k}}{\sum_{i_1 < \cdots < i_k} \left(D^{\mathbf{u}_{i_1} \cdots \mathbf{u}_{i_k}}_{i_1 \cdots i_k} \right)^2} \tag{11}$$

The superscript notation is borrowed from Doran/Lasenby, and denotes not a vector raised to a power, but this this special vector satisfying the following orthogonality and scaling criteria:

$$\mathbf{u}^k \cdot \mathbf{u}_i = \delta_{ki}. \tag{12}$$

Note that for $k = n - 1$, equation 11 reduces to

$$\mathbf{u}^n = \frac{D^{\mathbf{u}_1 \cdots \mathbf{u}_{n-1} \mathbf{e}}_{1 \cdots (n-1)}}{D^{\mathbf{u}_1 \cdots \mathbf{u}_n}_{1 \cdots n}}. \tag{13}$$

This or some other scaled version of this is likely as close as we can come to generalizing the cross product as an operation that takes vectors to vectors.

1.5 Example. $\mathbb{R}^3$ case. Perpendicular to two vectors.

Observe that for $\mathbb{R}^3$, writing $\mathbf{u} = \mathbf{u}_1$, $\mathbf{v} = \mathbf{u}_2$, $\mathbf{w} = \mathbf{u}_3$, and $\mathbf{w}' = \mathbf{u}_3{}^3$ this is:

$$\mathbf{w}' = \frac{\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{vmatrix}}{\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}} = \frac{\mathbf{u} \times \mathbf{v}}{(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}} \tag{14}$$

This is the cross product scaled by the (signed) volume for the parallelopiped spanned by the three vectors.

2 Derivation with GA.

Regression with respect to a set of vectors can be expressed directly. For vectors $\mathbf{u}_i$ write $\mathbf{B} = \mathbf{u}_1 \wedge \mathbf{u}_2 \cdots \mathbf{u}_k$. Then for any vector we have:

$$
\begin{aligned}
\mathbf{x} &= \mathbf{x}\mathbf{B}\frac{1}{\mathbf{B}} \\
&= \left\langle \mathbf{x}\mathbf{B}\frac{1}{\mathbf{B}} \right\rangle_1 \\
&= \left\langle (\mathbf{x}\cdot\mathbf{B} + \mathbf{x}\wedge\mathbf{B})\frac{1}{\mathbf{B}} \right\rangle_1
\end{aligned}
$$

All the grade three and grade five terms are selected out by the grade one operation, leaving just

$$
\mathbf{x} = (\mathbf{x}\cdot\mathbf{B})\cdot\frac{1}{\mathbf{B}} + (\mathbf{x}\wedge\mathbf{B})\cdot\frac{1}{\mathbf{B}}. \tag{15}
$$

This last term is the rejective component.

$$
\mathrm{Rej}_{\mathbf{B}}(\mathbf{x}) = (\mathbf{x}\wedge\mathbf{B})\cdot\frac{1}{\mathbf{B}} = \frac{(\mathbf{x}\wedge\mathbf{B})\cdot\mathbf{B}^\dagger}{\mathbf{B}\mathbf{B}^\dagger} \tag{16}
$$

Here we see in the denominator the squared sum of determinants in the denominator of equation 10:

$$
\mathbf{B}\mathbf{B}^\dagger = \sum_{i_1<\cdots<i_k} \left(D^{\mathbf{u}_{i_1}\cdots\mathbf{u}_{i_k}}_{i_1\cdots i_k}\right)^2
$$

In the numerator we have the dot product of two wedge products, each expressible as sums of determinants:

$$
\mathbf{B}^\dagger = (-1)^{k(k-1)/2} \sum_{i_1<\cdots<i_k} D^{\mathbf{u}_{i_1}\cdots\mathbf{u}_{i_k}}_{i_1\cdots i_k} \mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_k}
$$

And

$$
\mathbf{x}\wedge\mathbf{B} = \sum_{i_1<\cdots<i_{k+1}} D^{\mathbf{x}\mathbf{u}_{i_1}\cdots\mathbf{u}_{i_k}}_{i_1\cdots i_{k+1}} \mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_{k+1}}
$$

Dotting these is all the grade one components of the product. Performing that calculation would likely provide an explicit confirmation of the inductive hypothesis of equation 10. This can be observed directly for the $k+1=n$ case. That product produces a Laplace expansion sum.

$$
(\mathbf{x}\wedge\mathbf{B})\cdot\mathbf{B}^\dagger = D^{\mathbf{x}\mathbf{u}_1\cdots\mathbf{u}_{n-1}}_{12\cdots n}\left(\mathbf{e}_1 D^{\mathbf{u}_1\cdots\mathbf{u}_{n-1}}_{234\cdots n} - \mathbf{e}_2 D^{\mathbf{u}_1\cdots\mathbf{u}_{n-1}}_{134\cdots n} + \mathbf{e}_3 D^{\mathbf{u}_1\cdots\mathbf{u}_{n-1}}_{124\cdots n}\right)
$$

$$
(\mathbf{x}\wedge\mathbf{B})\cdot\frac{1}{\mathbf{B}} = \frac{D^{\mathbf{x}\mathbf{u}_1\cdots\mathbf{u}_{n-1}}_{12\cdots n} D^{\mathbf{e}\mathbf{u}_1\cdots\mathbf{u}_{n-1}}_{12\cdots n}}{\sum_{i_1<\cdots<i_k}\left(D^{\mathbf{u}_{i_1}\cdots\mathbf{u}_{i_k}}_{i_1\cdots i_k}\right)^2} \tag{17}
$$

Thus equation 10 for the $k = n - 1$ case is proved without induction. A proof for the $k + 1 < n$ case would be harder. No proof is required if one picks the set of basis vectors $\mathbf{e}_i$ such that $\mathbf{e}_i \wedge \mathbf{B} = 0$ (then the $k + 1 = n$ result applies). I believe that proves the general case too if one observes that a rotation to any other basis in the span of the set of vectors only changes the sign of the each of the determinants, and the product of the two sign changes will then have value one.

Follow through of the details for a proof of original non GA induction hypothesis is probably not worthwhile since this reciprocal frame vector problem can be tackled with a different approach using a subspace pseudovector.

It's notable that although this had no induction in the argument above, note that it is fundamentally required. That is because there is an inductive proof required to prove that the general wedge and dot product vector formulas:

$$\mathbf{x} \cdot \mathbf{B} = \frac{1}{2}(\mathbf{x}\mathbf{B} - (-1)^k \mathbf{B}\mathbf{x})$$

$$\mathbf{x} \wedge \mathbf{B} = \frac{1}{2}(\mathbf{x}\mathbf{B} + (-1)^k \mathbf{B}\mathbf{x})$$

from the GA axioms (that's an easier proof without the mass of indexes and determinant products.)

3 Pseudovector from rejection.

As noted in the previous section the reciprocal frame vector $\mathbf{u}^k$ is the vector in the direction of $\mathbf{u}_k$ that has no component in span $\mathbf{u}_1, \cdots, \mathbf{u}_{k-1}$, normalized such that $\mathbf{u}_k \cdot \mathbf{u}^k = 1$. Explicitly, with $\mathbf{B} = \mathbf{u}_1 \wedge \mathbf{u}_2 \cdots \wedge \mathbf{u}_{k-1}$ this is:

$$\mathbf{u}^k = \frac{(\mathbf{u}_k \wedge \mathbf{B}) \cdot \mathbf{B}}{\mathbf{u}_k \cdot ((\mathbf{u}_k \wedge \mathbf{B}) \cdot \mathbf{B})} \tag{18}$$

This is derived from equation 16, after noting that $\frac{\mathbf{B}^\dagger}{\mathbf{B}\mathbf{B}^\dagger} \propto \mathbf{B}$, and further scaling to produce the desired othornomal property of equation 12 that defines the reciprocal frame vector.

3.1 back to reciprocal result.

Now, equation 18 looks considerably different from the Doran/Lasenby result. Reduction to a direct pseudovector/blade product is possible since the dot product here can be converted to a direct product.

$$
\begin{aligned}
(\mathbf{u}_k \wedge \mathbf{B}) \cdot \mathbf{B} &= \underbrace{(\mathbf{x}\mathbf{B})}_{\mathbf{x}=\mathbf{u}_k-(\mathbf{u}_k\cdot\mathbf{B})\cdot\frac{1}{\mathbf{B}}} \cdot\mathbf{B} \\
&= \langle \mathbf{x}\mathbf{B}\mathbf{B}\rangle_1 \\
&= \mathbf{x}\mathbf{B}^2 \\
&= \left(\left(\mathbf{u}_k - (\mathbf{u}_k \cdot \mathbf{B}) \cdot \frac{1}{\mathbf{B}}\right) \wedge \mathbf{B}\right)\mathbf{B} \\
&= (\mathbf{u}_k \wedge \mathbf{B})\mathbf{B}
\end{aligned}
$$

Thus equation 18 is a scaled pseudovector for the subspace defined by span $\mathbf{u}_i$, multiplied by a k-1 blade.

4 Components of a vector.

The delta property of equation 12 allows one to use the reciprocal frame vectors and the basis that generated them to calculate the coordinates of the a vector with respect to this (not necessarily orthonormal) basis.

That's a pretty powerful result, but somewhat obscured by the Doran/Lasenby super/sub script notation.

Suppose one writes a vector in span $\mathbf{u}_i$ in terms of unknown coefficients

$$\mathbf{a} = \sum a_i \mathbf{u}_i$$

Dotting with $\mathbf{u}^j$ gives:

$$\mathbf{a} \cdot \mathbf{u}^j = \sum a_i \mathbf{u}_i \cdot \mathbf{u}^j = \sum a_i \delta_{ij} = a_j$$

Thus

$$\mathbf{a} = \sum (\mathbf{a} \cdot \mathbf{u}^i)\mathbf{u}_i \tag{19}$$

Similarly, writing this vectors in terms of $\mathbf{u}^i$ we have

$$\mathbf{a} = \sum b_i \mathbf{u}^i$$

Dotting with $\mathbf{u}_j$ gives:

$$\mathbf{a} \cdot \mathbf{u}_j = \sum b_i \mathbf{u}^i \cdot \mathbf{u}_j = \sum b_i \delta_{ij} = b_j$$

Thus

$$\mathbf{a} = \sum (\mathbf{a} \cdot \mathbf{u}_i)\mathbf{u}^i \tag{20}$$

We are used to seeing the equation for components of a vector in terms of a basis in the following form:

$$\mathbf{a} = \sum(\mathbf{a}\cdot\mathbf{u}_i)\mathbf{u}_i \tag{21}$$

This is true only when the basis vectors are orthonormal. Equations 19 and 20 provide the general decomposition of a vector in terms of a general linearly independent set.

4.1 Reciprocal frame vectors by solving coordinate equation.

A more natural way to these results are to take repeated wedge products. Given a vector decomposition in terms of a basis $\mathbf{u}_i$, we want to solve for a_i:

$$\mathbf{a} = \sum_{i=1}^{k} a_i \mathbf{u}_i$$

The solution, from the wedge is:

$$\begin{aligned}
\mathbf{a}\wedge(\mathbf{u}_1\wedge\mathbf{u}_2\cdots\check{\mathbf{u}}_i\cdots\wedge\mathbf{u}_k &= a_i(-1)^{i-1}\mathbf{u}_1\wedge\cdots\wedge\mathbf{u}_k \\
\implies a_i &= (-1)^{i-1}\frac{\mathbf{a}\wedge(\mathbf{u}_1\wedge\mathbf{u}_2\cdots\check{\mathbf{u}}_i\cdots\wedge\mathbf{u}_k}{\mathbf{u}_1\wedge\cdots\wedge\mathbf{u}_k})
\end{aligned}$$

The complete vector in terms of components is thus:

$$\mathbf{a} = \sum(-1)^{i-1}\frac{\mathbf{a}\wedge(\mathbf{u}_1\wedge\mathbf{u}_2\cdots\check{\mathbf{u}}_i\cdots\wedge\mathbf{u}_k)}{\mathbf{u}_1\wedge\cdots\wedge\mathbf{u}_k}\mathbf{u}_i \tag{22}$$

We are used to seeing the coordinates expressed in terms of dot products instead of wedge products. As in $\mathbb{R}^3$ where the pseudovector allows wedge products to be expressed in terms of the dot product we can do the same for the general case.

Writing $\mathbf{B}\in\bigwedge^{k-1}$ and $\mathbf{I}\in\bigwedge^{k}$ we want to reduce an equation of the following form

$$\frac{\mathbf{a}\wedge\mathbf{B}}{\mathbf{I}} = \frac{1}{\mathbf{I}}\frac{\mathbf{a}\mathbf{B}+(-1)^{k-1}\mathbf{B}\mathbf{a}}{2} \tag{23}$$

The pseudovector either commutes or anticommutes with a vector in the subspace depending on the grade

$$\begin{aligned}
\mathbf{I}\mathbf{a} &= \mathbf{I}\cdot\mathbf{a} + \underbrace{\mathbf{I}\wedge\mathbf{a}}_{=0} \\
&= (-1)^{k-1}\mathbf{a}\cdot\mathbf{I} \\
&= (-1)^{k-1}\mathbf{a}\mathbf{I}
\end{aligned}$$

Substituting back into equation 23 we have

$$
\begin{aligned}
\frac{\mathbf{a}\wedge\mathbf{B}}{\mathbf{I}} &= (-1)^{k-1}\frac{\mathbf{a}\left(\frac{1}{\mathbf{I}}\mathbf{B}\right)+\left(\frac{1}{\mathbf{I}}\mathbf{B}\right)\mathbf{a}}{2} \\
&= (-1)^{k-1}\mathbf{a}\cdot\left(\frac{1}{\mathbf{I}}\mathbf{B}\right) \\
&= \mathbf{a}\cdot\left(\mathbf{B}\frac{1}{\mathbf{I}}\right)
\end{aligned}
$$

With $\mathbf{I} = \mathbf{u}_1 \wedge \cdots \mathbf{u}_k$, and $\mathbf{B} = \mathbf{u}_1 \wedge \mathbf{u}_2 \cdots \check{\mathbf{u}}_i \cdots \wedge \mathbf{u}_k$, back substitution back into equation 22 is thus

$$
\mathbf{a} = \sum \mathbf{a}\cdot\left((-1)^{i-1}\mathbf{B}\frac{1}{\mathbf{I}}\right)\mathbf{u}_i
$$

The final result yields the reciprocal frame vector $\mathbf{u}^k$, and we see how to arrive at this result naturally attempting to answer the question of how to find the coordinates of a vector with respect to a (not necessarily orthonormal) basis.

$$
\mathbf{a} = \sum \mathbf{a}\cdot\underbrace{\left((\mathbf{u}_1\wedge\mathbf{u}_2\cdots\check{\mathbf{u}}_i\cdots\wedge\mathbf{u}_k)\frac{(-1)^{i-1}}{\mathbf{u}_1\wedge\cdots\wedge\mathbf{u}_k}\right)}_{\mathbf{u}^k}\mathbf{u}_i \tag{24}
$$

5 Components of a bivector.

To find the coordinates of a bivector with respect to an arbitrary basis we have a similar problem. For a vector basis $\mathbf{a}_i$, introduce a bivector basis $\mathbf{a}_i \wedge \mathbf{a}_j$, and write

$$
\mathbf{B} = \sum_{u<v} b_{uv}\mathbf{a}_u \wedge \mathbf{a}_v \tag{25}
$$

Wedging with $\mathbf{a}_i \wedge \mathbf{a}_j$ will select all but the ij component. Specifically

$$
\begin{aligned}
\mathbf{B}\wedge(\mathbf{a}_1\wedge\cdots\check{\mathbf{a}}_i\cdots\check{\mathbf{a}}_j\cdots\wedge\mathbf{a}_k) &= b_{ij}\mathbf{a}_i\wedge\mathbf{a}_j\wedge(\mathbf{a}_1\wedge\cdots\check{\mathbf{a}}_i\cdots\check{\mathbf{a}}_j\cdots\wedge\mathbf{a}_k) \\
&= b_{ij}(-1)^{j-2+i-1}(\mathbf{a}_1\wedge\cdots\wedge\mathbf{a}_k)
\end{aligned}
$$

Thus

$$
b_{ij} = (-1)^{i+j-3}\mathbf{B}\wedge\frac{(\mathbf{a}_1\wedge\cdots\check{\mathbf{a}}_i\cdots\check{\mathbf{a}}_j\cdots\wedge\mathbf{a}_k)}{\mathbf{a}_1\wedge\cdots\wedge\mathbf{a}_k} \tag{26}
$$

We want to put this in dot product form like equation 24. To do so we need a generalized grade reduction formula

$$(\mathbf{A}_a \wedge \mathbf{A}_b) \cdot \mathbf{A}_c = \mathbf{A}_a \cdot (\mathbf{A}_b \cdot \mathbf{A}_c) \tag{27}$$

This holds when $a + b \leq c$. Writing $\mathbf{A} = \mathbf{a}_1 \wedge \cdots \check{\mathbf{a}}_i \cdots \check{\mathbf{a}}_j \cdots \wedge \mathbf{a}_k$, and $\mathbf{I} = \mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k$, we have

$$\begin{aligned}
(\mathbf{B} \wedge \mathbf{A})\frac{1}{\mathbf{I}} &= (\mathbf{B} \wedge \mathbf{A}) \cdot \frac{1}{\mathbf{I}} \\
&= \mathbf{B} \cdot (\mathbf{A} \cdot \frac{1}{\mathbf{I}}) \\
&= \mathbf{B} \cdot (\mathbf{A}\frac{1}{\mathbf{I}})
\end{aligned}$$

Thus the bivector in terms of it's coordinates for this basis is:

$$\sum_{u<v} \mathbf{B} \cdot \left((\mathbf{a}_1 \wedge \cdots \check{\mathbf{a}}_u \cdots \check{\mathbf{a}}_v \cdots \wedge \mathbf{a}_k) \frac{(-1)^{u+v-2-1}}{\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k} \right) \mathbf{a}_u \wedge \mathbf{a}_v \tag{28}$$

It's easy to see how this generalizes to higher order blades since equation 27 is good for all required grades. In all cases, the form is going to be the same, with only differences in sign and the number of omitted vectors in the **A** blade.

For example for a trivector

$$\mathbf{T} = \sum_{u<v<w} t_{uvw} \mathbf{a}_u \wedge \mathbf{a}_v \wedge \mathbf{a}_w$$

It's pretty straightforward to show that this can be decomposed as follows

$$\mathbf{T} = \sum_{u<v<w} \mathbf{T} \cdot \left((\mathbf{a}_1 \wedge \cdots \check{\mathbf{a}}_u \cdots \check{\mathbf{a}}_v \cdots \check{\mathbf{a}}_w \cdots \wedge \mathbf{a}_k) \frac{(-1)^{u+v+w-3-2-1}}{\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k} \right) \mathbf{a}_u \wedge \mathbf{a}_v \wedge \mathbf{a}_w \tag{29}$$

5.1 Compare to GAFP.

Doran/Lasenby's GAFP demonstrates equation 24, and with some incomprehendable steps skips to a generalized result of the form [1]

$$\mathbf{B} = \sum_{i<j} \mathbf{B} \cdot \left(\mathbf{a}^j \wedge \mathbf{a}^i \right) \mathbf{a}_i \wedge \mathbf{a}_j \tag{30}$$

GAFP states this for general multivectors instead of bivectors, but the idea is the same.

[1] In retrospect I don't think that the in between steps had anything to do with logical sequence. The authors wanted some of the results for subsequent stuff (like: rotor recovery) and sandwiched it between the vector and reciprocal frame multivector results somewhat out of sequence.

This makes intuitive sense based on the very similar vector result. This doesn't show that the generalized reciprocal frame k-vectors calculated in equation 28 or equation 29 can be produced simply by wedging the corresponding individual reciprocal frame vectors.

To show that either takes algebraic identities that I do not know, or am not thinking of as applicable. Alternately perhaps it would just take simple brute force.

Easier is to demonstrate the validity of the final result directly. Then assuming my direct calculations are correct implicitly demonstrates equivalence.

Starting with $\mathbf{B}$ as defined in equation 25, take dot products with $\mathbf{a}^j \wedge \mathbf{a}^i$.

$$
\begin{aligned}
\mathbf{B} \cdot (\mathbf{a}^j \wedge \mathbf{a}^i) &= \sum_{u<v} b_{uv} (\mathbf{a}_u \wedge \mathbf{a}_v) \cdot (\mathbf{a}^j \wedge \mathbf{a}^i) \\
&= \sum_{u<v} b_{uv} \begin{vmatrix} \mathbf{a}_u \cdot \mathbf{a}^i & \mathbf{a}_u \cdot \mathbf{a}^j \\ \mathbf{a}_v \cdot \mathbf{a}^i & \mathbf{a}_v \cdot \mathbf{a}^j \end{vmatrix} \\
&= \sum_{u<v} b_{uv} \begin{vmatrix} \delta_{ui} & \delta_{uj} \\ \delta_{vi} & \delta_{vj} \end{vmatrix}
\end{aligned}
$$

Consider this determinant when $u = i$ for example

$$
\begin{vmatrix} \delta_{ui} & \delta_{uj} \\ \delta_{vi} & \delta_{vj} \end{vmatrix} = \begin{vmatrix} 1 & \delta_{ij} \\ \delta_{vi} & \delta_{vj} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \delta_{vi} & \delta_{vj} \end{vmatrix} = \delta_{vj}
$$

If any one index is common, then both must be common ($ij = uv$) for this determinant to have a non-zero (ie: one) value. On the other hand, if no index is common then all the δ's are zero.

Like equation 12 this demonstrates an orthonormal selection behavior like the reciprocal frame vector. It has the action:

$$
(\mathbf{a}_i \wedge \mathbf{a}_j) \cdot (\mathbf{a}^v \wedge \mathbf{a}^u) = \delta_{ij,uv} \tag{31}
$$

This means that we can write b_{uv} directly in terms of a bivector dot product

$$
b_{uv} = \mathbf{B} \cdot (\mathbf{a}^v \wedge \mathbf{a}^u)
$$

and thus proves equation 30. Proof of the general result also follows from the determinant expansion of the respective blade dot products.

5.2 Direct expansion of bivector in terms of reciprocal frame vectors

Looking at linear operators I realized that the result for bivectors above can follow more easily from direct expansion of a bivector written in terms of vector factors:

$$
\begin{aligned}
\mathbf{a} \wedge \mathbf{b} &= \sum (\mathbf{a} \cdot \mathbf{u}_i \mathbf{u}^i) \wedge (\mathbf{b} \cdot \mathbf{u}_j \mathbf{u}^j) \\
&= \sum_{i<j} \left(\mathbf{a} \cdot \mathbf{u}_i \mathbf{b} \cdot \mathbf{u}_j - \mathbf{a} \cdot \mathbf{u}_j \mathbf{b} \cdot \mathbf{u}_i \right) \mathbf{u}^i \wedge \mathbf{u}^j \\
&= \sum_{i<j} \begin{vmatrix} \mathbf{a} \cdot \mathbf{u}_i & \mathbf{a} \cdot \mathbf{u}_j \\ \mathbf{b} \cdot \mathbf{u}_i & \mathbf{b} \cdot \mathbf{u}_j \end{vmatrix} \mathbf{u}^i \wedge \mathbf{u}^j
\end{aligned}
$$

When the set of vectors $\mathbf{u}_i = \mathbf{u}^i$ are othornormal we've already calculated this result when looking at the wedge product in a differential forms context:

$$
\mathbf{a} \wedge \mathbf{b} = \sum_{i<j} \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix} \mathbf{u}_i \wedge \mathbf{u}_j \tag{32}
$$

For this general case for possibly non-orthonormal frames, this determinant of dot products can be recognized as the dot product of two blades

$$
\begin{aligned}
(\mathbf{a} \wedge \mathbf{b}) \cdot (\mathbf{u}_j \wedge \mathbf{u}_i) &= \mathbf{a} \cdot (\mathbf{b} \cdot (\mathbf{u}_j \wedge \mathbf{u}_i)) \\
&= \mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{u}_j \mathbf{u}_i - \mathbf{b} \cdot \mathbf{u}_i \mathbf{u}_j) \\
&= \mathbf{b} \cdot \mathbf{u}_j \mathbf{a} \cdot \mathbf{u}_i - \mathbf{b} \cdot \mathbf{u}_i \mathbf{a} \cdot \mathbf{u}_j
\end{aligned}
$$

Thus we have a decomposion of the bivector directly into a sum of components for the reciprocal frame bivectors:

$$
\mathbf{a} \wedge \mathbf{b} = \sum_{i<j} \left((\mathbf{a} \wedge \mathbf{b}) \cdot (\mathbf{u}_j \wedge \mathbf{u}_i) \right) \mathbf{u}^i \wedge \mathbf{u}^j \tag{33}
$$