Expanding the grade 0 part of a multivector product.

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One can show that the grade zero component of a multivector product is independent of the order of the terms:

$$\langle \mathbf{AB} \rangle_0 = \langle \mathbf{BA} \rangle_0 \tag{1}$$

Doran/Lasenby has an elegant proof of this, but a dumber proof using an explicit expansion by basis also works and highlights the similarities with the standard component definition of the vector dot product.

Writing:

$$\mathbf{A} = \sum_i \langle \mathbf{A} \rangle_i$$
$$\mathbf{B} = \sum_i \langle \mathbf{B} \rangle_i$$

The product of $\mathbf{A}$ and $\mathbf{B}$ is:

$$\begin{aligned}
\mathbf{AB} &= \sum_{ij} \langle \mathbf{A} \rangle_i \langle \mathbf{B} \rangle_j \\
&= \sum_{ij} \sum_{k=0}^{\min(i,j)} \Big\langle \langle \mathbf{A} \rangle_i \langle \mathbf{B} \rangle_j \Big\rangle_{2k+|i-j|}
\end{aligned}$$

$$\mathbf{AB} = \sum_{ij} \sum_{k=0}^{\min(i,j)} \Big\langle \langle \mathbf{A} \rangle_i \langle \mathbf{B} \rangle_j \Big\rangle_{2k+|i-j|} \tag{2}$$

To get a better feel for this, consider an example

$$\mathbf{A} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_{12} + \mathbf{e}_{13} + \mathbf{e}_{34} + \mathbf{e}_{345}$$
$$\mathbf{B} = \mathbf{e}_2 + \mathbf{e}_{21} + \mathbf{e}_{23}$$

$$\mathbf{AB} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_{12} + \mathbf{e}_{13} + \mathbf{e}_{34} + \mathbf{e}_{345})(\mathbf{e}_2 + \mathbf{e}_{21} + \mathbf{e}_{23})$$

Here are multivectors with grades ranging from zero to three. This multiplication will include vector/vector, vector/bivector, vector/trivector, bivector/bivector, and bivector/trivector. Some of these will be grade lowering, some grade preserving and some grade raising.

Only the like grade terms can potentially generate grade zero terms, so the grade zero terms of the product in equation 2 are:

$$\mathbf{AB} = \sum_{i=j} \left\langle \langle \mathbf{A} \rangle_i \langle \mathbf{B} \rangle_j \right\rangle_0 \tag{3}$$

Using the example above we have

$$\langle \mathbf{AB} \rangle_0 = \langle (\mathbf{e}_1 + \mathbf{e}_2)\mathbf{e}_2 \rangle_0 + \langle (\mathbf{e}_{12} + \mathbf{e}_{13} + \mathbf{e}_{34})\mathbf{e}_{21} \rangle_0$$

In general one can introduce an orthonormal basis $\sigma^k = \{\sigma^k_i\}_i$ for each of the $\langle\rangle_k$ spaces. Here orthonormal is with respect to the k-vector dot product

$$\sigma^k_i \cdot \sigma^k_j = (-1)^{k(k-1)/2} \delta_{ij} \tag{4}$$

then one can decompose each of the k-vectors with respect to that basis:

$$\langle \mathbf{A} \rangle_k = \sum_i \left(\langle \mathbf{A} \rangle_k \cdot \sigma^k_i \right) \frac{1}{\sigma^k_i}$$

$$\langle \mathbf{B} \rangle_k = \sum_j \left(\langle \mathbf{B} \rangle_k \cdot \sigma^k_j \right) \frac{1}{\sigma^k_j}$$

Thus the scalar part of the product is

$$\begin{aligned}
\langle \mathbf{AB} \rangle_0 &= \sum_{k,i,j} \left\langle \left(\langle \mathbf{A} \rangle_k \cdot \sigma^k_i \right) \frac{1}{\sigma^k_i} \left(\langle \mathbf{B} \rangle_k \cdot \sigma^k_j \right) \frac{1}{\sigma^k_j} \right\rangle_0 \\
&= \sum_{k,i,j} \left\langle \sigma^k_i \sigma^k_j \right\rangle_0 \left(\langle \mathbf{A} \rangle_k \cdot \sigma^k_i \right) \left(\langle \mathbf{B} \rangle_k \cdot \sigma^k_j \right) \\
&= \sum_{k,i,j} (-1)^{k(k-1)/2} \delta_{ij} \left(\langle \mathbf{A} \rangle_k \cdot \sigma^k_i \right) \left(\langle \mathbf{B} \rangle_k \cdot \sigma^k_j \right)
\end{aligned}$$

Thus the complete scalar product can be written

$$\langle \mathbf{AB} \rangle_0 = \sum_{k,i} (-1)^{k(k-1)/2} \left(\langle \mathbf{A} \rangle_k \cdot \sigma^k_i \right) \left(\langle \mathbf{B} \rangle_k \cdot \sigma^k_i \right) \tag{5}$$

Note, compared to the vector dot product, the alternation in sign, which is dependent on the grades involved.

Also note that this now trivially proves that the scalar product is commutative.

Perhaps more importantly we see how similar this generalized dot product is to the standard component formulation of the vector dot product we are so used to. At a glance the componentless geometric algebra formulation seems so much different than the standard vector dot product expressed in terms of components, but we see here that this is in fact not the case.