Notes on shear transformation.

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[Dorst et al.(2007)Dorst, Fontijne, and Mann] and [Doran and Lasenby(2003)] both give examples of shear transformations of the following form:

$$F(a) = a + \alpha(a \cdot f)g.$$

[Doran and Lasenby(2003)] uses this to compute the determinant without putting the operator in matrix form. They end up stating that

$$F(A) = A + \alpha(A \cdot f) \wedge g \tag{1}$$

holds for any grade blade A. For grade 1 that is true since

$$\begin{aligned}
(a \cdot f)g &= \langle (a \cdot f)g \rangle_1 \\
&= (a \cdot f) \wedge g.
\end{aligned}$$

They demonstrate equation 1 holds for the grade 2 case. To me it seems like an induction is required to make their statement for any grade.

Question: Is there some other principle that I didn't notice in my reading that allows assertion of equation 1 for any grade blade without the induction.

1.1 Proof for any grade blade.

For $A \in \bigwedge^r$

$$\begin{aligned}
F(A) \wedge F(b) &= (A + \alpha(A \cdot f) \wedge g) \wedge (b + \alpha(b \cdot f)g) \\
&= A \wedge b + \alpha \left((A \cdot f) \wedge g \wedge b + (b \cdot f) A \wedge g \right) \\
&= A \wedge b + \alpha \underbrace{\left(-(A \cdot f) \wedge b + A(b \cdot f) \right)}_{(*)} \wedge g \\
&= A \wedge b + \alpha((A \wedge b) \cdot f) \wedge g
\end{aligned}$$

QED.

Verification below of $(*) = f \cdot (A \wedge b)$ is required to complete the proof (can probably find that in one of the books or papers but it is derivable easily enough).

Do I have a sign mixup here somewhere? Now that I look again I see that GAFP has the result in different order $(A \cdot f) \wedge g$, and I get negation reconciling the two.

1.2 Dot product reduction of blade by one.

$$
\begin{aligned}
f \cdot (A \wedge b) &= \frac{1}{2} \langle f(A \wedge b) \rangle_r \\
&= \frac{1}{2} \langle f(Ab + (-1)^r bA) \rangle_r \\
&= \frac{1}{2} \langle (fA)b + (-1)^r fbA \rangle_r \\
&= \frac{1}{2} \langle ((-1)^r Af + 2f \cdot A)b + (-1)^r fbA \rangle_r \\
&= (f \cdot A) \wedge b + \frac{(-1)^r}{2} \langle Afb + fbA \rangle_r \\
&= (f \cdot A) \wedge b + (-1)^r (f \cdot b)A + \frac{(-1)^r}{2} \langle Af \wedge b + f \wedge bA \rangle_r
\end{aligned}
$$

This last term, the symmetric product of a bivector with a blade is zero. The grades $r-2, r+4, \ldots$ terms are symmetric, and the other grades $r, r+4, \ldots$ are antisymmetric.

Thus we have

$$
f \cdot (A \wedge b) = (f \cdot A) \wedge b - (-1)^r (f \cdot b)A \tag{2}
$$

This generalizes the familiar vector reduction formula to higher grades. Observe that for the vector case we need the most general definition of the wedge product for the scalar-vector wedge product (grade $1-0$ part of the product).

References

[Doran and Lasenby(2003)] C. Doran and A.N. Lasenby. *Geometric algebra for physicists*. Cambridge University Press New York, 2003.

[Dorst et al.(2007)Dorst, Fontijne, and Mann] L. Dorst, D. Fontijne, and S. Mann. Geometric Algebra for Computer Science, 2007.