Revisit Stokes derivation.

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Sept 27, 2008. Last Revision: $Date : 2008/11/1514 : 15 : 02$

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1 Algebraic description of oriented boundaries.

Having used pictorial methods to enumerate the bounding loop and area elements [Joot()] in the previous derivation of the vector and bivector forms of Stokes's, makes the application of these formulas harder. Here this will be revisited, with the aim of remedying this, as well as obtaining a proof for the general case, which was not possible because of a lack of exactly this algebraic formulation.

1.1 Parallelogram parameterization.

An oriented curve around a parallelogram in $\mathbb{R}^n$ is illustrated in figure 1. We want to evaluate the line integral around this path

$$\oint \mathbf{f} \cdot d\mathbf{r} = \int du_1 \, \mathbf{f} \cdot \frac{\partial \mathbf{r}}{\partial u_1}\bigg|_{u_2(0)}^{u_2(1)} - \int du_2 \, \mathbf{f} \cdot \frac{\partial \mathbf{r}}{\partial u_2}\bigg|_{u_1(0)}^{u_1(1)} \tag{1}$$

$$\tag{2}$$

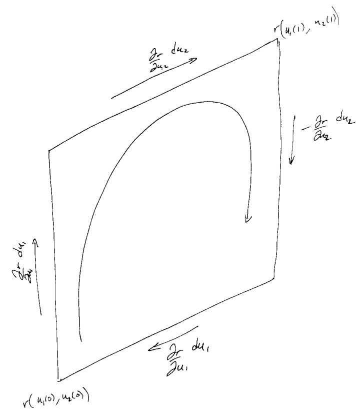

Figure 1: Two variable parameterization of $\mathbb{R}^n$ parallelogram

Now, we can put this in a more symmetric form utilizing a reciprocal frame to enumerate the alternation. Write

$$
\begin{aligned}
\mathbf{r}_{u_i} &= \frac{\partial \mathbf{r}}{\partial u_i} \\
\mathbf{r}^{u_j} \cdot \mathbf{r}_{u_i} &= {\delta^j}_i \\
I &= \mathbf{r}_{u_1} \wedge \mathbf{r}_{u_2} \\
I\mathbf{r}^{u_1} &= I \cdot \mathbf{r}^{u_1} = -\mathbf{r}_{u_2} \\
I\mathbf{r}^{u_2} &= I \cdot \mathbf{r}^{u_2} = \mathbf{r}_{u_1}.
\end{aligned}
$$

We don't care to actually calculate the reciprocal frame vectors. They just work well to describe the alternation in terms of the pseudoscalar for the plane.

Substituting back into 1 we have

$$
\oint \mathbf{f} \cdot d\mathbf{r} = \int du_1 \, \mathbf{f} \cdot (I\mathbf{r}^{u_2})\Big|_{u_2(0)}^{u_2(1)} + \int du_2 \, \mathbf{f} \cdot (I\mathbf{r}^{u_1})\Big|_{u_1(0)}^{u_1(1)}
$$

Or

$$
\oint \mathbf{f} \cdot d\mathbf{r} = \sum_i \int \frac{du_1 du_2}{du_i} \, \mathbf{f} \cdot (I\mathbf{r}^{u_i})\Big|_{u_i(0)}^{u_i(1)} \tag{3}
$$

This completes the goal of expressing the line integral in a fashion that doesn't require drawing any pictures, and gives a hint about how to do the same for general $\bigwedge^k \mathbb{R}^n$ case.

As before this can be written in terms of its integrals

$$\begin{aligned}
\oint \mathbf{f} \cdot d\mathbf{r} &= \sum_{i,j \neq i} \int_{u_j(0)}^{u_j(1)} du_j \int_{u_i(0)}^{u_i(1)} \frac{\partial}{\partial u_i} \mathbf{f} \cdot (I\mathbf{r}^{u_i}) du_i \\
&= \iint du_1 du_2 \sum \frac{\partial}{\partial u_i} \mathbf{f} \cdot (I\mathbf{r}^{u_i})
\end{aligned}$$

Evaluating the derivatives to prove the Stokes/Green's result will be deferred for now (may instead proving the general case once formulated).

1.2 Parallelopiped parameterization.

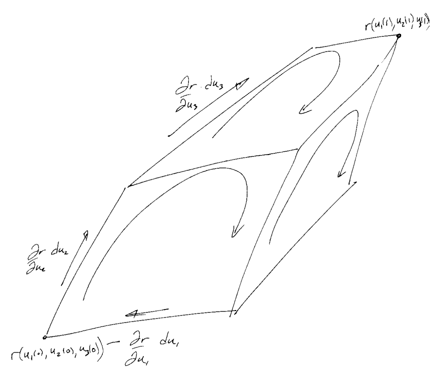

Figure 2: Three variable parameterization of $\mathbb{R}^n$ parallelopiped

Next, lets evaluate the bivector area dot products, as in figure 2.

$$
\begin{aligned}
\oiint \mathbf{F} \cdot d\mathbf{A} &= \iint du_2 du_1 \; F \cdot \left(\mathbf{r}_{u_2} \wedge \mathbf{r}_{u_1}\right)\Big|_{u_3(1)}^{u_3(0)} \\
&+ \iint du_3 du_1 \; F \cdot \left(\mathbf{r}_{u_3} \wedge \mathbf{r}_{u_1}\right)\Big|_{u_2(0)}^{u_2(1)} \\
&+ \iint du_3 du_2 \; F \cdot \left(\mathbf{r}_{u_3} \wedge -\mathbf{r}_{u_2}\right)\Big|_{u_1(0)}^{u_1(1)}
\end{aligned}
$$

Again introducing reciprocal vectors to enumerate the alternation, but now write I as a pseudoscalar for the parallelopiped subspace that the area bounds

$$
\begin{aligned}
I &= \mathbf{r}_{u_1} \wedge \mathbf{r}_{u_2} \wedge \mathbf{r}_{u_3} \\
I\mathbf{r}^{u_1} &= \mathbf{r}_{u_2} \wedge \mathbf{r}_{u_3} \\
I\mathbf{r}^{u_2} &= -\mathbf{r}_{u_1} \wedge \mathbf{r}_{u_3} \\
I\mathbf{r}^{u_3} &= \mathbf{r}_{u_1} \wedge \mathbf{r}_{u_2}
\end{aligned}
$$

Substituting we have a form almost identical to the line integral of equation 3.

$$
\oiint \mathbf{F} \cdot d\mathbf{A} = \sum \iint \frac{du_1 du_2 du_3}{du_i} \; F \cdot \left(I\mathbf{r}^{u_i}\right)\Big|_{u_i(0)}^{u_i(1)} \tag{4}
$$

$$
= \iiint du_1 du_2 du_3 \sum \frac{\partial}{\partial u_i} F \cdot \left(I\mathbf{r}^{u_i}\right) \tag{5}
$$

1.3 General case.

Having found that the line integral and oriented area integrals can be expressed uniformly in the same algebraic form, it is reasonable to define an integral with such structure as a directed hypervolume boundary for any grade blade, and then verify that this yields the expected generalized Stokes result that has been proven for only the vector and area cases.

Writing

$$
\begin{aligned}
F &\in \bigwedge^{k-1} \mathbb{R}^n \\
d^k\mathbf{x} &= \frac{\partial \mathbf{r}}{\partial u_1} \wedge \frac{\partial \mathbf{r}}{\partial u_2} \wedge \cdots \wedge \frac{\partial \mathbf{r}}{\partial u_k} du_1 du_2 \cdots du_k = I du_1 du_2 \cdots du_k
\end{aligned}
$$

We wish to prove the general Stokes equation for a hyper-parallelepiped volume

$$
\int_V (\nabla \wedge F) \cdot d^k\mathbf{x} = \int_{\partial V} F \cdot d^{k-1}\mathbf{x} \tag{6}
$$

With the presumption that this will algebraically be identical to the line integral and area integral cases for vectors and bivectors respectively we want to evaluate

$$\begin{aligned}
\int_{\partial V} F \cdot d^{k-1}\mathbf{x} &= \sum \int \frac{du_1 du_2 \cdots du_k}{du_i} \; F \cdot (I\mathbf{r}^{u_i})\Big|_{u_i(0)}^{u_i(1)} \\
&= \int_V du_1 du_2 \cdots du_k \sum \frac{\partial}{\partial u_i} F \cdot (I\mathbf{r}^{u_i})
\end{aligned}$$

$$\int_{\partial V} F \cdot d^{k-1}\mathbf{x} = \int_V du_1 du_2 \cdots du_k \sum \frac{\partial F}{\partial u_i} \cdot (I\mathbf{r}^{u_i}) + F \cdot \left(\frac{\partial}{\partial u_i} I\mathbf{r}^{u_i}\right). \tag{7}$$

The last term here sums to zero. The messy long proof of this can be found at the end. Assuming that proven this leaves us with the following identity

$$\int_{\partial V} F \cdot d^{k-1}\mathbf{x} = \int_V du_1 du_2 \cdots du_k \sum \frac{\partial F}{\partial u_i} \cdot (I\mathbf{r}^{u_i}) \tag{8}$$

We wish to show that this equals

$$\int_V du_1 du_2 \cdots du_k (\nabla \wedge F) \cdot I,$$

after which point we have both formulated algebraically the boundary integral, and proven the general $k-1$blade Stokes theorem of equation 6.

1.4 Is a coordinate free proof possible?

Note that

$$\begin{aligned}
\frac{\partial F}{\partial u_i} \cdot (I\mathbf{r}^{u_i}) &= \left\langle \frac{\partial F}{\partial u_i} I\mathbf{r}^{u_i} \right\rangle_0 \\
&= \left\langle \mathbf{r}^{u_i} \frac{\partial F}{\partial u_i} I \right\rangle_0 \\
&= \left(\mathbf{r}^{u_i} \wedge \frac{\partial F}{\partial u_i}\right) \cdot I
\end{aligned}$$

Can the reduction of this wedge product to curl form be done without coordinates? It would also be fairly easy to go in circles here since the reciprocal frame vectors can be calculated in terms of the pseudoscalar I.

1.5 Notation for coordinate expansion.

I didn't have any luck finding a coordinate free way as outlined above to prove the general result. The dumb brute force way is still possible though, expanding both sides and comparing.

The following will be used in the sections below

$$
\begin{aligned}
\mathbf{r} &= \gamma_j x^j \\
\mathbf{r}_{u_i} &= \gamma_j \frac{\partial x^j}{\partial u_i} \\
I\mathbf{r}^{u_i} &= (-1)^{k-i} \mathbf{r}_{u_1} \wedge \mathbf{r}_{u_2} \wedge \cdots \widehat{\mathbf{r}_{u_i}} \cdots \wedge \mathbf{r}_{u_k}
\end{aligned}
$$

$$
I\mathbf{r}^{u_i} = (-1)^{k-i} \gamma_{j_1} \wedge \cdots \gamma_{j_{k-1}} \frac{\partial x^{j_1}}{\partial u_1} \cdots \widehat{\frac{\partial}{\partial u_i}} \cdots \frac{\partial x^{j_{k-1}}}{\partial u_k} \tag{9}
$$

Here the overhat is used to indicate omission.

1.6 Expanding the curl dot by coordinates.

One half of the comparision will based on the expansion of $(\nabla \wedge F) \cdot I$. We calculate

$$
\begin{aligned}
F &= \frac{1}{(k-1)!} F_{j_1 j_2 \cdots j_{k-1}} \gamma^{j_1} \wedge \gamma^{j_2} \cdots \wedge \gamma^{j_{k-1}} \\
I &= \gamma_{m_1} \wedge \gamma_{m_2} \cdots \wedge \gamma_{m_k} \frac{\partial x^{m_1}}{\partial u_1} \frac{\partial x^{m_2}}{\partial u_2} \cdots \frac{\partial x^{m_k}}{\partial u_k} \\
\nabla &= \gamma^{j_k} \frac{\partial}{\partial x^{j_k}}
\end{aligned}
$$

$$
\begin{aligned}
\nabla \wedge F &= \frac{1}{(k-1)!} \frac{\partial F_{j_1 j_2 \cdots j_{k-1}}}{\partial x^{j_k}} \gamma^{j_k} \wedge \gamma^{j_1} \wedge \gamma^{j_2} \cdots \wedge \gamma^{j_{k-1}} \\
&= \frac{(-1)^{k-1}}{(k-1)!} \frac{\partial F_{j_1 j_2 \cdots j_{k-1}}}{\partial x^{j_k}} \gamma^{j_1} \wedge \gamma^{j_2} \cdots \wedge \gamma^{j_{k-1}} \wedge \gamma^{j_k}.
\end{aligned}
$$

Now, put this all together

$$
\begin{aligned}
(\nabla \wedge F) \cdot I = & \frac{(-1)^{k-1}}{(k-1)!} \left(\gamma^{j_1} \wedge \gamma^{j_2} \cdots \wedge \gamma^{j_k}\right) \cdot (\gamma_{m_1} \wedge \gamma_{m_2} \cdots \wedge \gamma_{m_k}) \\
& \frac{\partial F_{j_1 j_2 \cdots j_{k-1}}}{\partial x^{j_k}} \frac{\partial x^{m_1}}{\partial u_1} \frac{\partial x^{m_2}}{\partial u_2} \cdots \frac{\partial x^{m_k}}{\partial u_k} \\
&= \frac{(-1)^{k-1}}{(k-1)!} \delta^{j_k}{}_{m_1} \delta^{j_{k-1}}{}_{m_2} \cdots \delta^{j_1}{}_{m_k} \epsilon^{m_1 m_2 \cdots m_k} \frac{\partial F_{j_1 j_2 \cdots j_{k-1}}}{\partial x^{j_k}} \frac{\partial x^{m_1}}{\partial u_1} \frac{\partial x^{m_2}}{\partial u_2} \cdots \frac{\partial x^{m_k}}{\partial u_k} \\
&= \frac{(-1)^{k-1}}{(k-1)!} \epsilon^{m_1 m_2 \cdots m_k} \frac{\partial F_{m_k m_{k-1} \cdots m_2}}{\partial x^{m_1}} \frac{\partial x^{m_1}}{\partial u_1} \frac{\partial x^{m_2}}{\partial u_2} \cdots \frac{\partial x^{m_k}}{\partial u_k}
\end{aligned}
$$

Now, to reverse a k vector, or its corresponding antisymmetric tensor as above we have to perform the following number of swaps

$$k-1+k-2+\cdots+1 = k(k-1)/2$$

We can use this to tidy the indexes above

$$k-1+(k-1)(k-2)/2 = k(k-1)/2,$$

and thus write

$$(\nabla \wedge F) \cdot I = \frac{(-1)^{k(k-1)/2}}{(k-1)!} \epsilon^{m_1 m_2 \cdots m_k} \frac{\partial F_{m_2 \cdots m_k}}{\partial x^{m_1}} \frac{\partial x^{m_1}}{\partial u_1} \frac{\partial x^{m_2}}{\partial u_2} \cdots \frac{\partial x^{m_k}}{\partial u_k} \tag{10}$$

1.7 Expanding the boundary integral by coordinates.

The remainder of the proof is to verify that the expression 10 matches the differential form in equation 8.

To do so we have to expand

$$\frac{\partial F}{\partial u_i} \cdot (I \mathbf{r}^{u_i})$$

$$\frac{\partial F}{\partial u_i} = \frac{\partial x^{m_1}}{\partial u_i} \frac{\partial}{\partial x^{m_1}} \frac{1}{(k-1)!} F_{m_2 m_3 \cdots m_k} \gamma^{m_2} \wedge \gamma^{m_2} \cdots \wedge \gamma^{m_k}$$

Dotting this with equation 9 we have

$$\begin{aligned}
\frac{\partial F}{\partial u_i} \cdot (I \mathbf{r}^{u_i}) &= \frac{(-1)^{k-i}}{(k-1)!} \frac{\partial F_{m_2 m_3 \cdots m_k}}{\partial x^{m_1}} (\gamma^{m_2} \wedge \gamma^{m_2} \cdots \wedge \gamma^{m_k}) \cdot (\gamma_{j_1} \wedge \cdots \gamma_{j_{k-1}}) \\
&\quad \frac{\partial x^{j_1}}{\partial u_1} \cdots \widehat{\frac{\partial}{\partial u_i}} \cdots \frac{\partial x^{j_{k-1}}}{\partial u_k} \frac{\partial x^{m_1}}{\partial u_i} \\
&= \frac{(-1)^{k-i}}{(k-1)!} \frac{\partial F_{m_2 m_3 \cdots m_k}}{\partial x^{m_1}} \delta^{m_k}{}_{j_1} \delta^{m_{k-1}}{}_{j_2} \cdots \delta^{m_2}{}_{j_{k-1}} \epsilon^{j_1 j_2 \cdots j_{k-1}} \\
&\quad \frac{\partial x^{j_1}}{\partial u_1} \cdots \widehat{\frac{\partial}{\partial u_i}} \cdots \frac{\partial x^{j_{k-1}}}{\partial u_k} \frac{\partial x^{m_1}}{\partial u_i} \\
&= \frac{(-1)^{k-i}}{(k-1)!} \frac{\partial F_{m_2 m_3 \cdots m_k}}{\partial x^{m_1}} \epsilon^{m_k m_{k-1} \cdots m_2} \frac{\partial x^{m_k}}{\partial u_1} \cdots \widehat{\frac{\partial}{\partial u_i}} \cdots \frac{\partial x^{m_2}}{\partial u_k} \frac{\partial x^{m_1}}{\partial u_i} \\
&= \frac{-(-1)^{i+k(k-1)/2}}{(k-1)!} \frac{\partial F_{m_2 m_3 \cdots m_k}}{\partial x^{m_1}} \epsilon^{m_2 \cdots m_{k-1}} \frac{\partial x^{m_k}}{\partial u_1} \cdots \widehat{\frac{\partial}{\partial u_i}} \cdots \frac{\partial x^{m_2}}{\partial u_k} \frac{\partial x^{m_1}}{\partial u_i}
\end{aligned}$$

After nicely arranging the m_i indexes to match equation 10, the partials don't match. Perhaps about a change of variables:

$$
\begin{aligned}
m_1 &= n_i \\
m_k &= n_1 \\
m_{k-1} &= n_2 \\
&\cdots\cdots \\
m_2 &= n_k
\end{aligned}
$$

(with appropriate adjustments for i=1)

$$
\begin{aligned}
\frac{\partial F}{\partial u_i} \cdot (I\mathbf{r}^{u_i}) &= \frac{-(-1)^{i+k(k-1)/2}}{(k-1)!} \frac{\partial F_{n_k n_{k-1} \cdots \widehat{n_i} \cdots n_1}}{\partial x^{n_i}} \epsilon^{n_k \cdots \widehat{n_i} \cdots n_1} \frac{\partial x^{n_1}}{\partial u_1} \frac{\partial x^{n_2}}{\partial u_2} \cdots \frac{\partial x^{n_k}}{\partial u_k} \\
&= \frac{-(-1)^{i+k(k-1)/2}}{(k-1)!} \frac{\partial F_{n_1 n_2 \cdots \widehat{n_i} \cdots n_k}}{\partial x^{n_i}} \epsilon^{n_1 \cdots \widehat{n_i} \cdots n_k} \frac{\partial x^{n_1}}{\partial u_1} \frac{\partial x^{n_2}}{\partial u_2} \cdots \frac{\partial x^{n_k}}{\partial u_k}
\end{aligned}
$$

Here we have the product of two completely antisymmetric tensors, both with the same set of indexes, so any alternation of those indexes has no effect. The only sign changes come from the $-(-1)^i$ coefficient.

To verify consistency with equation 10 it remains to prove that within the sum the following two are identical

$$
\epsilon^{m_1 m_2 \cdots m_k} \frac{\partial F_{m_2 \cdots m_k}}{\partial x^{m_1}} \tag{11}
$$

$$
(-1)^{i+1} \frac{\partial F_{n_1 n_2 \cdots \widehat{n_i} \cdots n_k}}{\partial x^{n_i}} \epsilon^{n_1 \cdots \widehat{n_i} \cdots n_k}. \tag{12}
$$

Examination and a bit of thought shows this to be the case. FIXME: this statement is intuition based, and I'm having trouble describing exactly why I say so. Revisit this later (for now I'd rather spend the time working with the result than to complete the last details of the proof).

2 Summary.

Summarizing, a proof has been given for the general multivector Stokes equation, that provides equivalent volume and boundary integral expressions

$$
\int_V (\nabla \wedge F) \cdot d^k\mathbf{x} = \int_{\partial V} F \cdot d^{k-1}\mathbf{x}. \tag{13}
$$

The proof of this result was restricted to a hyper-parallelopiped volume and its corresponding boundary. Additional arguments are required to extend this to arbitrary shapes. That argument follows the loop integral case where cancellation of oppositely oriented surfaces in adjacent volumes can be used to build up an arbitrary shape in terms of small parallelopiped volumes.

In addition to the proof of this result, a specific algebraic (non-pictorial) meaning has been given to the boundary differential form $d^{k-1}\mathbf{x}$. We have used the following notation

$$\mathbf{r}_{u_i} = \frac{\partial \mathbf{r}}{\partial u_i} \tag{14}$$

$$\mathbf{r}^{u_i} \cdot \mathbf{r}_{u_j} = \delta^i{}_j \tag{15}$$

$$I = \mathbf{r}_{u_1} \wedge \mathbf{r}_{u_2} \wedge \cdots \wedge \mathbf{r}_{u_k} \tag{16}$$

$$= \gamma_{m_1} \wedge \gamma_{m_2} \cdots \wedge \gamma_{m_k} \frac{\partial x^{m_1}}{\partial u_1} \frac{\partial x^{m_2}}{\partial u_2} \cdots \frac{\partial x^{m_k}}{\partial u_k} \tag{17}$$

$$\tag{18}$$

$$\begin{aligned}
I\mathbf{r}^{u_i} &= I \cdot \mathbf{r}^{u_i} \\
&= (-1)^{k-i} \mathbf{r}_{u_1} \wedge \mathbf{r}_{u_2} \wedge \cdots \widehat{\mathbf{r}_{u_i}} \cdots \wedge \mathbf{r}_{u_k} \\
&= (-1)^{k-i} \gamma_{j_1} \wedge \cdots \gamma_{j_{k-1}} \frac{\partial x^{j_1}}{\partial u_1} \cdots \widehat{\frac{\partial}{\partial u_i}} \cdots \frac{\partial x^{j_{k-1}}}{\partial u_k}
\end{aligned}$$

Here $\mathbf{r}$, as parameterized by u_i spans the hyper-parallopiped, and $\mathbf{r}(u_1, \cdots, u_i(1), \cdots)$, and $\mathbf{r}(u_1, \cdots, u_i(0), \cdots)$ represent boundaries of the surface with respect to parameter u_i. Putting things together we have the following algebraic description of the boundary

$$\int_{\partial V} F \cdot d^{k-1}\mathbf{x} = \sum \int du_1 \cdots \widehat{du_i} \cdots du_k \; F \cdot (I\mathbf{r}^{u_i})\Big|_{u_i(0)}^{u_i(1)} \tag{19}$$

Observe that we have a Jacobian like relationship above due to the alternation provided by the wedge product. For this reason it would make sense to introduce vector differentials

$$d\mathbf{x}_i = \frac{\partial \mathbf{r}}{\partial u_i} du_i, \tag{20}$$

in order to suppress the explicit parameterization.

$$\int_{\partial V} F \cdot d^{k-1}\mathbf{x} = \int \sum (-1)^{k-i} \; F \cdot \left(d\mathbf{x}_1 \wedge \cdots \widehat{d\mathbf{x}_i} \cdots \wedge d\mathbf{x}_k \right)\Big|_{u_i(0)}^{u_i(1)} \tag{21}$$

$$= \int \sum (-1)^{k-i} \; F \cdot \left(d\mathbf{x}_1 \wedge \cdots \widehat{d\mathbf{x}_i} \cdots \wedge d\mathbf{x}_k \right)\Big|_{\partial \mathbf{x}_i} \tag{22}$$

In the LHS of equation 13 we also have a specific meaning for the k-vector volume element

$$d^k\mathbf{x} = I du_1 du_2 \cdots du_k \tag{23}$$

$$= d\mathbf{x}_1 \wedge d\mathbf{x}_2 \cdots \wedge d\mathbf{x}_k \tag{24}$$

Also notable for the volume integral is its tensor formulation, where we have the volume jacobian determinant explicitly

$$\int_V (\nabla \wedge F) \cdot d^k\mathbf{x} = \int_V \frac{(-1)^{k(k-1)/2}}{(k-1)!} \frac{\partial F_{m_2 \cdots m_k}}{\partial x^{m_1}} \frac{\partial(x^{m_1}, \cdots, x^{m_k})}{\partial(u_1, \cdots, u_k)} du_1 \cdots du_k \tag{25}$$

The other interesting thing worth noting is the reciprocal expression for curl projected onto the integration subspace

$$(\nabla \wedge F) \cdot I = \left(\sum_i \mathbf{r}^{u_i} \wedge \frac{\partial F}{\partial u_i} \right) \cdot I \tag{26}$$

I wasn't able to use this, but having mostly completed the proof, this is proved as a side effect. Here mostly means that the unsatisfactory treatments (really handwaving) marked with FIXMEs should be revisited to consider this multivector form of Stokes theorem fully proved here.

3 Messy proof of zero sum in equation 7

Here is the deferred proof that the sum of the differentials of the area elements are zero

$$\int du_1 du_2 \cdots du_k \sum F \cdot \left(\frac{\partial}{\partial u_i} I \mathbf{r}^{u_i} \right).$$

Although not elegant, the partials here can be expanded by coordinates as done in the previous line and area proofs.

We want to prove that

$$\sum (-1)^{k-i} \gamma_{j_1} \wedge \cdots \gamma_{j_{k-1}} \frac{\partial}{\partial u_i} \frac{\partial x^{j_1}}{\partial u_1} \cdots \widehat{\frac{\partial}{\partial u_i}} \cdots \frac{\partial x^{j_{k-1}}}{\partial u_k} = 0 \tag{27}$$

as was done previously in the vector and bivector cases. Pick as an example the $i = 3$ case, and assume that $k > 2$ since the two simpler cases have been proven explicitly. For that i, we have the following terms

$$\sum(-1)^{k-3}\gamma_{j_1}\wedge\cdots\gamma_{j_{k-1}}\Bigg(\frac{\partial^2 x^{j_1}}{\partial u_3 \partial u_1}\frac{\partial x^{j_2}}{\partial u_2}\frac{\partial x^{j_3}}{\partial u_4}\cdots\frac{\partial x^{j_{k-1}}}{\partial u_k}$$
$$+\frac{\partial^2 x^{j_2}}{\partial u_3 \partial u_2}\frac{\partial x^{j_1}}{\partial u_1}\frac{\partial x^{j_3}}{\partial u_4}\cdots\frac{\partial x^{j_{k-1}}}{\partial u_k}$$
$$+\frac{\partial^2 x^{j_3}}{\partial u_3 \partial u_4}\frac{\partial x^{j_1}}{\partial u_1}\frac{\partial x^{j_2}}{\partial u_2}\cdots\frac{\partial x^{j_{k-1}}}{\partial u_k}$$
$$+\cdots$$
$$+\frac{\partial^2 x^{j_k}}{\partial u_3 \partial u_k}\frac{\partial x^{j_1}}{\partial u_1}\frac{\partial x^{j_2}}{\partial u_2}\cdots\frac{\partial x^{j_{k-2}}}{\partial u_{k-1}}\Bigg)$$

Picking any mixed partial term we expect cancellation with the opposing mixed partial. Two representive values of i should be sufficient to see that the sum is zero. First pick $i = 1$, so that $(-1)^{k-3} = (-1)^{k-1}$, and look at the matching partial for the $\frac{\partial^2}{\partial u_1 \partial u_3}$ term above

$$\sum(-1)^{k-1}\gamma_{j_1}\wedge\cdots\gamma_{j_{k-1}}\Bigg(\frac{\partial^2 x^{j_1}}{\partial u_1 \partial u_2}\frac{\partial x^{j_2}}{\partial u_3}\frac{\partial x^{j_3}}{\partial u_4}\cdots\frac{\partial x^{j_{k-1}}}{\partial u_k}$$
$$+\frac{\partial^2 x^{j_2}}{\partial u_1 \partial u_3}\frac{\partial x^{j_1}}{\partial u_2}\frac{\partial x^{j_3}}{\partial u_4}\cdots\frac{\partial x^{j_{k-1}}}{\partial u_k}$$
$$+\cdots$$
$$+\frac{\partial^2 x^{j_k}}{\partial u_1 \partial u_k}\frac{\partial x^{j_1}}{\partial u_2}\frac{\partial x^{j_2}}{\partial u_3}\cdots\frac{\partial x^{j_{k-2}}}{\partial u_{k-1}}\Bigg).$$

Swapping dummy indexes j_1 and j_2 here one can see that the $\frac{\partial^2}{\partial u_1 \partial u_3}$ and $\frac{\partial^2}{\partial u_3 \partial u_1}$ terms cancel.

Now pick $i = 2$, so that $(-1)^{k-1} = -(-1)^{k-2}$, and look at the matching partial for the $\frac{\partial^2}{\partial u_1 \partial u_2}$ term above.

$$\sum(-1)^{k-2}\gamma_{j_1}\wedge\cdots\gamma_{j_{k-1}}\Bigg(\frac{\partial^2 x^{j_1}}{\partial u_2 \partial u_1}\frac{\partial x^{j_2}}{\partial u_3}\frac{\partial x^{j_3}}{\partial u_4}\cdots\frac{\partial x^{j_{k-1}}}{\partial u_k}$$
$$+\cdots$$
$$+\frac{\partial^2 x^{j_k}}{\partial u_2 \partial u_k}\frac{\partial x^{j_1}}{\partial u_1}\frac{\partial x^{j_2}}{\partial u_3}\cdots\frac{\partial x^{j_{k-2}}}{\partial u_{k-1}}\Bigg).$$

No swap of indexes is required and we see again that the mixed partials cancel. Now this is perhaps a slightly lazy proof, but working with indexes in the abstract without assigning specific numbers gets confusing. It is clear to me that the end result will be a zero sum for this term.

FIXME: A cleanup of this proof should be possible to eliminate the special case comparisions above. The tough part is simply writing all the terms in a manipulatable fashion. Then procede to split the sum into terms that differ by even and odd separation of indexes. Summing over indexes greater and indexes lesser, then swapping indexes as appropriate should complete the proof. Alternatively, perhaps I'll figure out a clever way later to demonstrate this more directly without resorting to this messy coordinate expansion.

References

[Joot()] Peeter Joot. Vector and bivector integral relationships. "http://sites.google.com/site/peeterjoot/geometric-algebra/vector_integral_relations.pdf".