Wedge product norm and GA bivector norm comparision.

Peeter Joot

Jan 2008 or so.

Informative to look at the bivector formed by two unit perpendicular vectors in a plane.

$$
\begin{aligned}
\hat{\mathbf{u}} \wedge \frac{(\mathbf{v} - (\hat{\mathbf{u}} \cdot \mathbf{v})\, \hat{\mathbf{u}})}{\left(\frac{1}{\|\mathbf{u}\|^2} \sum_{i<j} \left(D^{\mathbf{uv}}_{ij} \right)^2 \right)^{1/2}}
&= \frac{\mathbf{u} \wedge \mathbf{v}}{\left(\sum_{i<j} \left(D^{\mathbf{uv}}_{ij} \right)^2 \right)^{1/2}} \\
&= \frac{\mathbf{u} \wedge \mathbf{v}}{\|\mathbf{u} \wedge \mathbf{v}\|}
\end{aligned}
$$

Here, $\|bivector\|$ is taken with the most obvious definition. For an arbitrary bivector we define:

$$
\|\sum_{i<j} g_{ij} \hat{\mathbf{e}}_i \wedge \hat{\mathbf{e}}_j\|^2 = \sum_{i<j} {g_{ij}}^2
$$

Let's see how this compares with the GA norm (square) of a bivector.

$$
\begin{aligned}
(\mathbf{u} \wedge \mathbf{v})^2 &= (\mathbf{uv} - \mathbf{u} \cdot \mathbf{v})^2 \\
&= (\mathbf{uv} - \mathbf{u} \cdot \mathbf{v})(\mathbf{uv} - \mathbf{u} \cdot \mathbf{v}) \\
&= \mathbf{uvuv} - 2(\mathbf{u} \cdot \mathbf{v})\mathbf{uv} + (\mathbf{u} \cdot \mathbf{v})^2
\end{aligned}
$$

Since $\mathbf{u} \cdot \mathbf{v} = (\mathbf{uv} + \mathbf{vu})/2$, then $\mathbf{vu} = 2\mathbf{u} \cdot \mathbf{v} - \mathbf{uv}$.

$$
\begin{aligned}
(\mathbf{u} \wedge \mathbf{v})^2 &= \mathbf{u}(2\mathbf{u}\cdot\mathbf{v} - \mathbf{u}\mathbf{v})\mathbf{v} - 2(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\mathbf{v} + (\mathbf{u}\cdot\mathbf{v})^2) \\
&= -\mathbf{u}\mathbf{u}\mathbf{v}\mathbf{v} + (\mathbf{u}\cdot\mathbf{v})^2 \\
&= (\mathbf{u}\cdot\mathbf{v})^2 - \|\mathbf{u}\|^2\|\mathbf{v}\|^2 \\
&= -\sum_{i<j}\left(D^{\mathbf{u}\mathbf{v}}_{ij}\right)^2 \\
&= -\|\mathbf{u}\wedge\mathbf{v}\|^2
\end{aligned}
$$

So, we can also define the norm of a bivector in terms of its GA product:

$$
\|\mathbf{u}\wedge\mathbf{v}\|^2 = -(\mathbf{u}\wedge\mathbf{v})^2
$$

And, with implications to Rotors, we can see that the role of the GA product $\mathbf{I} = \hat{\mathbf{e}}_i\hat{\mathbf{e}}_j, i \neq j$ can also be filled by the wedge product of any two perpendicular unit vectors in a plane, or equivalently any unit bivector. So, we have -1 as the GA magnitude (square) of any unit bivector:

$$
\left(\frac{\mathbf{u}\wedge\mathbf{v}}{\|\mathbf{u}\wedge\mathbf{v}\|}\right)^2 = -1
$$

0.1 Perpendicular to vector in direction of second expressed as GA product.

Starting with the component of the vector $\mathbf{v}$ that is perpendicular to $\mathbf{u}$, we have:

$$
\mathbf{v}' = (\mathbf{v} - (\hat{\mathbf{u}}\cdot\mathbf{v})\,\hat{\mathbf{u}}) = \frac{1}{\|\mathbf{u}\|^2}\left(\|\mathbf{u}\|^2\mathbf{v} - \mathbf{u}\,(\mathbf{u}\cdot\mathbf{v})\right)
$$

Using the GA product $\mathbf{u}^2 = \|\mathbf{u}\|^2$, and $\mathbf{u}\wedge\mathbf{v} = \mathbf{u}\mathbf{v} - \mathbf{u}\cdot\mathbf{v}$,

$$
\Rightarrow \mathbf{v}' = \frac{\mathbf{u}}{\mathbf{u}^2}\left(\mathbf{u}\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\right) = \frac{1}{\mathbf{u}}\,(\mathbf{u}\wedge\mathbf{v}) = \hat{\mathbf{u}}\,(\hat{\mathbf{u}}\wedge\mathbf{v})
$$

Thus we can write the decomposition of the vector $\mathbf{v}$ into components parallel and perpendicular to $\mathbf{u}$ as:

$$
\mathbf{v} = \hat{\mathbf{u}}\,(\hat{\mathbf{u}}\cdot\mathbf{v}) + \hat{\mathbf{u}}\,(\hat{\mathbf{u}}\wedge\mathbf{v})
$$

Or,

$$
\mathbf{v} = \frac{1}{\mathbf{u}}\,(\mathbf{u}\cdot\mathbf{v}) + \frac{1}{\mathbf{u}}\,(\mathbf{u}\wedge\mathbf{v})
$$

Without the GA product formulation we didn't have a way without messy determinant sums to formulate the perpendicular component of this vector.