Tensor Derivation of Covariant Lorentz Force from Lagrangian

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1 Motivation.

In [Joot(b)], and before that in [Joot(a)] Clifford algebra derivations of the STA form of the covariant Lorentz force equation were derived. As an exercise in tensor manipulation try the equivalent calculation using only tensor manipulation.

2 Calculation.

The starting point will be an assumed Lagrangian of the following form

$$\mathcal{L} = \frac{1}{2}v^2 + (q/m)A \cdot v/c \tag{1}$$

$$= \frac{1}{2}\dot{x}_\alpha \dot{x}^\alpha + (q/mc)A_\beta \dot{x}^\beta \tag{2}$$

Here v is the proper (four)velocity, and A is the four potential. And following [Doran and Lasenby(2003)], we use a positive time signature for the metric tensor $(+---)$.

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = (q/mc)\frac{\partial A_\beta}{\partial x^\mu}\dot{x}^\beta$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} &= \frac{\partial}{\partial \dot{x}^\mu} \left(\frac{1}{2} g_{\alpha\beta} \dot{x}^\beta \dot{x}^\alpha \right) + (q/mc) \frac{\partial (A_\alpha \dot{x}^\alpha)}{\partial \dot{x}^\mu} \\
&= \frac{1}{2} \left(g_{\alpha\mu} \dot{x}^\alpha + g_{\mu\beta} \dot{x}^\beta \right) + (q/mc) A_\mu \\
&= \dot{x}_\mu + (q/mc) A_\mu
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x^\mu} &= \frac{d}{d\tau} \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \\
(q/mc) \frac{\partial A_\beta}{\partial x^\mu} \dot{x}^\beta &= \ddot{x}_\mu + (q/mc) \dot{x}^\beta \frac{\partial A_\mu}{\partial x^\beta} \\
\implies & \\
\ddot{x}_\mu &= (q/mc) \dot{x}^\beta \left(\frac{\partial A_\beta}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\beta} \right) \\
&= (q/mc) \dot{x}^\beta \left(\partial_\mu A_\beta - \partial_\beta A_\mu \right)
\end{aligned}
$$

This is

$$
m \ddot{x}_\mu = (q/c) F_{\mu\beta} \dot{x}^\beta \tag{3}
$$

The wikipedia article [wikipedia()] writes this in the equivalent indexes toggled form

$$
m \ddot{x}^\mu = (q/c) \dot{x}_\beta F^{\mu\beta}
$$

[Schiller()] (22nd edition, equation 467) writes this with the Maxwell tensor in mixed form

$$
b^\mu = \frac{q}{m} F_\nu{}^\mu u^\nu
$$

where b^μ is a proper acceleration. If one has to put the Lorentz equation it in tensor form, using a mixed index tensor seems like the nicest way since all vector quantities then have consistently placed indexes. Observe that he has used units with $c = 1$, and by comparison must also be using a time negative metric tensor.

3 Compare for reference to GA form.

To verify that this form is identical to familiar STA Lorentz Force equation,

$$\dot{p} = q(F \cdot v/c) \tag{4}$$

reduce this equation to coordinates. Starting with the RHS (leaving out the q/c)

$$\begin{aligned}
(F \cdot v) \cdot \gamma_\mu &= \frac{1}{2} F_{\alpha\beta} \dot{x}^\nu ((\gamma^\alpha \wedge \gamma^\beta) \cdot \gamma_\nu) \cdot \gamma_\mu \\
&= \frac{1}{2} F_{\alpha\beta} \dot{x}^\nu \left(\gamma^\alpha (\gamma^\beta \cdot \gamma_\nu) - \gamma^\beta (\gamma^\alpha \cdot \gamma_\nu) \right) \cdot \gamma_\mu \\
&= \frac{1}{2} \left(F_{\alpha\nu} \dot{x}^\nu \gamma^\alpha - F_{\nu\beta} \dot{x}^\nu \gamma^\beta \right) \cdot \gamma_\mu \\
&= \frac{1}{2} \left(F_{\mu\nu} \dot{x}^\nu - F_{\nu\mu} \dot{x}^\nu \right) \\
&= F_{\mu\nu} \dot{x}^\nu
\end{aligned}$$

And for the LHS

$$\begin{aligned}
\dot{p} \cdot \gamma_\mu &= m \ddot{x}_\alpha \gamma^\alpha \cdot \gamma_\mu \\
&= m \ddot{x}_\mu
\end{aligned}$$

Which gives us

$$m \ddot{x}_\mu = (q/c) F_{\mu\nu} \dot{x}^\nu \tag{5}$$

in agreement with 3.

References

[Doran and Lasenby(2003)] C. Doran and A.N. Lasenby. *Geometric algebra for physicists*. Cambridge University Press New York, 2003.

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