Some GR Notes.

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Contents

1 Motivation.

Some General relativity notes exploring ideas from emails with Lut Mentz. Prior to this I was under the impression that I had zero knowledge of GR, but it turns out that many of the ideas are really action based. Allowing the spacetime unit vectors to vary, something we are free to do in SR or newtonian mechanics too, results in a more general metric in a purely kinetic Lagrangian. This metric variation can be interpretted as a mechanism for introducing more general accererations very similar to fictious forces that one sees in a rotating frame or other non-uniform coordinate system.

These notes contain my attempt to walk through some of these ideas, to see if I can coherently explain them to myself. If I can't do so then I don't understand things sufficently. Being able to produce such an explaination may not mean that I truely understand the issues, but it is a required first step.

Useful references are Lut's writeup [Mentz()], and the schwarzchild calculation [Unknown()] from the online text reflections on relativity.

FIXME: Update here when Lut puts his Rindler metric derivation online.

First emails with Lut was about Lagrangian mass variation, where he was investigating the similarities between varying mass directly and the spatial/metric variation of GR. Some of the math bits of this can be found in [Joot(b)], but the exploritory physics bits he was attempting are probably more interesting.

1.1 Lagrangian for General Relativity.

The equations of motion resulting from a purely kinetic Lagrangian

$$\mathcal{L} = \frac{1}{2} \sum g_{bc}(q^a) \dot{q}^b \dot{q}^c, \tag{1}$$

can be found to be

$$0 = \ddot{q}^a + \dot{q}^b \dot{q}^c {\Gamma^a}_{bc} \tag{2}$$

$${\Gamma^a}_{bc} = \frac{1}{2} g^{ad} \left(\frac{\partial g_{cd}}{\partial q^b} + \frac{\partial g_{db}}{\partial q^c} - \frac{\partial g_{bc}}{\partial q^d} \right) \tag{3}$$

One such derivation can be found in the solution of problem one [Joot(c)], associated with the Lagrangian problem set for Dr. David Tong's online mechanics text [Tong()].

This is the Lagrangian for general relativity, once the metric tensor g_{ab} is specified.

1.2 General kinetic Lagrangian for fixed frame vectors.

One doesn't have to go to GR to find Kinetic energy expressions of the form in equation 1.

A simple example of a more general Kinetic energy description can be found by any use of non-orthonormal basis vectors, say $\{\mathbf{e}_i\}$, for the space.

Given such a non-orthonormal frame, the trick to calculating the coordinates is tied to an alternate set of basis vectors, called the reciprocal frame. Provided the initial set of vectors spans the space, one can always calculate this second pair such that they meet the following relationships:

$$\mathbf{e}^i \cdot \mathbf{e}_j = {\delta^i}_j$$

Calculating these reciprocal frame vectors is a linear algebra problem, essentially requiring a matrix inversion. Here we just assume they can be calculated and use them as a convienent way to find the coordinates in this non-orthogonal frame

$$
\begin{aligned}
\mathbf{x} &= \sum \mathbf{e}_j a_j \\
\mathbf{x} \cdot \mathbf{e}^i &= \sum (\mathbf{e}_j a_j) \cdot \mathbf{e}^i = \sum \delta^j{}_i a_j = a_i \\
\implies & \\
\mathbf{x} &= \sum \mathbf{e}_j (\mathbf{x} \cdot \mathbf{e}^j)
\end{aligned}
$$

It is customary to write $a_i = \mathbf{x} \cdot \mathbf{e}^i = x^i$, in order to have mixed upper and lower indexes for implied summation.

$$
\mathbf{x} = \sum \mathbf{e}_j x^j = \mathbf{e}_j x^j
$$

Once one has a way of calculating coordinates for an arbitrary basis, other quantities such as velocity can then be calculated

$$
\mathbf{v}^2 = \dot{\mathbf{x}} \cdot \dot{\mathbf{x}} = (\mathbf{e}_j \cdot \mathbf{e}_k) \dot{x}^j \dot{x}^k
$$

This $\mathbf{e}_j \cdot \mathbf{e}_k$ coefficient of the coordinates gets a special name, the metric tensor $g_{jk} = \mathbf{e}_j \cdot \mathbf{e}_k$. It is a symmetric and invertable quantity and can be employed to express the Kinetic Energy term of a single particle Lagrangian in the general tensor form

$$
\frac{1}{2} m \mathbf{v}^2 = \frac{1}{2} m g_{jk} \dot{x}^j \dot{x}^k
$$

Now, in this case g_{jk} is not a function of the coordinates (ie: of position) as in equation 1.

1.3 General kinetic expression for moving frame vectors.

Now, in the GR case the metric varies (or may) with position. What do we need to observe this same form in Newtonian physics? The immediate thing that comes to mind is the use of a curvalinear basis, where the basis vectors for a space are allowed to vary direction with position along some path. Initially that seemed reasonable to me but for some arbitrary parameterized path, wouldn't the metric then also vary with the path parameterization? If that was the case, the Lagrangian in equation 1 does not have the form of a kinetic energy expression.

To resolve this I considered an example. I can lay out two directions in my backyard, one along the vegetable garden parallel to the house roughly pointing north, and another diagonally across to my gate. This logically defines a coordinate system or set of frame vectors that I can make local measurements with respect to. Now, translation of this coordinate system to my dad's house 40 km to the south won't be a particularily logical for measuring there. He lives on a very steep hill.

I can pace out distances in my backyard without having to consider the curvature of the Earth and my dad can do the same for a long stretch of the hill walking up the street towards his house. The local frame vectors can be considered to lie along a flat surface if that surface area is small enough. Alternately we can say the associated metric associated with a surface coordinate system for a point on surface of the Earth can be considered constant for small enough measurements.

Now, to express the same ideas mathematically, consider a curve expressed parametrically between two points, such that all points along the path take the values

$$\mathbf{x}(\lambda) = x^i(\lambda)\mathbf{e}_i(\mathbf{x})$$

The vector distance between two points on this path is

$$\mathbf{x}(\lambda_2) - \mathbf{x}(\lambda_1) = x^i(\lambda_2)\mathbf{e}_i(\mathbf{x}(\lambda_2)) - x^i(\lambda_1)\mathbf{e}_i(\mathbf{x}(\lambda_1))$$

but this is the direct difference in position between these two points, not the distance along the curve. To be a true measure of the distance the difference in position has to also be small enough that the frame vectors lie in the same direction at both points to some approximation.

Given such an approximation one can then write

$$\begin{aligned}
\mathbf{x}(\lambda_2) - \mathbf{x}(\lambda_1) &= \left(x^i(\lambda_2) - x^i(\lambda_1)\right)\mathbf{e}_i(\mathbf{x}) \\
d\mathbf{x} &= \frac{dx^i}{d\lambda}\mathbf{e}_i(\mathbf{x})d\lambda
\end{aligned}$$

For such a representation to be valid, the variation of $\mathbf{e}_i$ at the point $\mathbf{x}$ has to be small enough that $\mathbf{e}_i$ can be considered constant. This is still not a very well defined statement mathematically, and it is not too hard to imagine scenerios where it totally fails. An example is a fractal like curve, something continuous but not differtiable at any point.

Assuming a sufficently differentiable curve then the distance along the curve between two points can be obtained from the integral

$$\begin{aligned}
ds &= \int_{\lambda_1}^{\lambda_2} \sqrt{\left(\frac{d\mathbf{x}}{d\lambda}\right)^2} d\lambda \\
&= \int_{\lambda_1}^{\lambda_2} \sqrt{g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}} d\lambda
\end{aligned}$$

Or

$$\left(\frac{ds}{d\lambda}\right)^2 = g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}$$

Now, what is the most natural parameterization? For physical situations time comes to mind, but if the particle stops for a while on the path, then this derivative goes to zero for a while even if the curve is continuous and has derivatives of all orders at all points. Falling back to the most simple curve as a motivator, the circle, use of fractions θ of the total circumference of the circle 2π naturally parameterizes points on the curve. The same thing can be done for any curve in Euclidean space, using arc length to parameterize a path. In terms of the metric tensor this is

$$\left(\frac{ds}{ds}\right)^2 = 1 = g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}$$

Introducing proper time for spacetime event path parameterization will be related to these spatial ideas. Before considering proper time some thought about general metrics for spacetime seems to be in order.

2 Metrics for spacetime in special relativity.

It was stated above that equation 1 was the Lagrangian for general relativity, once the metric is specified.

It is not at all obvious to me what physically motivates the choice of metric. In the two examples that Lut has given me, the Rindler metric and the Schwartzchild metric, this appears to be rather arbitrary.

2.1 Minkowski metric justification from the wave equation.

Picking the Minkowski metric for special relativity is not at all an obvious choice. An introductory physics book like [Lewis(1965)] chooses to introduce the idea indirectly by requiring that a coordinate transformation used to express the equation for a spherically expanding light shell

$$\sum_i (x^i)^2 - c^2 t^2 = \sum_i (x^{i'})^2 - c^2 t'^2 \tag{4}$$

is identical in any spacetime coordinate system. This expresses the idea that the speed of light is the same in any frame of reference and that both time and distance are to be measured only locally. The Lorentz transformation can then be derived from this notion of spherical shell equation invariance, and the Minkowski mixed signature metric can be observed to fundamentally describe both the Lorentz transformation and the shell invariance.

First reading this it was not at all obvious to me that 4 was a reasonable starting point. The constancy of the speed of light is easy to say, but saying it doesn't mean that the implications are understood.

Suppose that one is considering the standard relativistic scenerio of a fast spaceship ($v = c/2$ say), with headlights on. Is it that obvious that the speed of light percieved by an observer of the rocket will be c, or will it be $1.5c$?

This non-obvious nature is likely reflected by popular disbelief of relativity in the general non-physics student/professional population. The twin paradox, length contraction, and time dialation ideas are all very far from our general familarity and experience, yet these are the aspects of relativity that are popularized. What you don't see in Sci-Fi is the intrinsic connection between relativistic ideas and electromagnetism. The same people who may say they don't believe in relativity wouldn't say that they don't believe in their cell phone, DVD player, personal computer, or television, yet the physics and engineering behind all of these and relativity are the same!

My personal justification for the Minkowski metric comes from consideration of the wave nature of electromagnetism [Joot(a)]. A bit of study of electromagnetism shows that Maxwell's equations for electric and magnetic fields requires that

$$\left(-\boldsymbol{\nabla}^2 + \frac{1}{c^2}\frac{\partial^2}{\partial^2 t}\right)\mathbf{E} = 0$$
$$\left(-\boldsymbol{\nabla}^2 + \frac{1}{c^2}\frac{\partial^2}{\partial^2 t}\right)\mathbf{B} = 0$$

Electromagnetic radiation (light) appears to be a fundamentally wavelike phenomina. That is an idea that I think we are all fairly comfortable with. Mathematically, the mechanical introduction of an arbitrary change of variables $x^\mu \rightarrow x'^\mu$ should not change that. If one describes all the types of change of variables such that the wave equation retains its form in the second coordinate system, then application of the chain rule for the coordinate transformation essentially imposes a Lorentz transformation constraint on this transformation.

The approach of using the wave equation as a motivator requires some calculus, and thus still isn't something that is good for a Layman's justification of the Lorentz transformation and the corresponding Minkowski metric. Einstein himself actually has a nice Layman's treatment of special relativity in his book [Einstein(2005)]. The appendix of this book also has a simple derivation of the Lorentz transformation in two variables that is well worth reading.

2.2 Minkowski metric justification by taking vector square roots of Delambertian.

Also related to the wave equation one can justify the Minkowski metric by factoring the scalar wave equation (Delambertian) operator

$$-\boldsymbol{\nabla}^2 + \frac{1}{c^2}\frac{\partial^2}{\partial^2 t} = -\sum_i \frac{\partial^2}{\partial^2 x^i} + \frac{1}{c^2}\frac{\partial^2}{\partial^2 t}$$

into a vector product. If one writes the spacetime vector (with $x^0 = ct$) as

$$x = \gamma_\mu x^\mu$$

A spacetime gradient operator can be defined that squares to ± 1 times the wave equation operator

$$\nabla = \gamma^\mu \frac{\partial}{\partial x^\mu}$$

For the square of this operator to equal the wave equation operator we need two conditions

$$\begin{aligned} \gamma^\mu \cdot \gamma^\nu &= \delta^{\mu\nu}(\gamma^\mu)^2 \\ (\gamma^0)^2(\gamma^i)^2 &= -1. \end{aligned}$$

With these we can then write the wave equation operator as

$$-\boldsymbol{\nabla}^2 + \frac{1}{c^2}\frac{\partial^2}{\partial^2 t} = \frac{1}{(\gamma^0)^2}\nabla^2$$

There are an infinite number of ways to factor this scalar equation into a vector product, depending on the spacetime coordinate systems used, but all of them vary by a Lorentz transformation!

Requiring that an orthonormal spacetime basis has a Minkowski metric signature $(\gamma^0)^2(\gamma^i)^2 = -1$ has value independent of any mechanics and relativistic consideration. If nothing else the fact that this can be utilized to summarize Maxwell's equations as the clifford algebra product [Doran and Lasenby(2003)]

$$\nabla F = J/c\epsilon_0$$

Here ∇ defined as above, and we have a bivector for the field, a vector for the current and charge density, and the four space pseudoscalar ties the electric and magnetic field components together into a single complex number like quanity:

$$
\begin{aligned}
F &= \mathbf{E} + Ic\mathbf{B} \\
J &= c\rho\gamma_0 + J^i\gamma_i \\
I &= \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3
\end{aligned}
$$

The spatial vectors we are used to calculating with are expressed as spacetime bivectors

$$
\begin{aligned}
\sigma_i &= \gamma_i \wedge \gamma_0 \\
\sigma_i \cdot \sigma_j &= \delta_{ij} \\
\mathbf{E} &= E^i\sigma_i \\
\mathbf{B} &= B^i\sigma_i \\
\mathbf{J} &= J^i\sigma_i,
\end{aligned}
$$

where the bivector basis $\{\sigma_i\}$ for the spacetime split behave in all respects like Euclidean vectors.

If nothing else the Minkowski metric idea that is at the root of all of this has got visible value as it brings together all of electromagnetism under a single umbrella.

3 Utilizing the Minkowski metric

Presuming that some acceptable justification or motivation for the Minkowski metric has been accepted (I've outlined mine in the two sections above), this allows for expression of four-vector distances when an orthonormal spacetime basis is used.

3.1 Proper time and event arc length in SR.

The simplest example is for uniform motion along γ_1, relative to a fixed point x^1. Here the event path for a particle can be written

$$
x = ct\gamma_0 + (x^1 - vt)\gamma_1
$$

The differential distance for some parameterization of time $t = t(\lambda)$ is thus

$$\begin{aligned}\frac{dx}{d\lambda} &= \frac{dt}{d\lambda}\left(c\gamma_0 - v\gamma_1\right) \\ \left(\frac{dx}{d\lambda}\right)^2 = \frac{dx}{d\lambda}\cdot\frac{dx}{d\lambda} &= \left(\frac{dt}{d\lambda}\right)^2 \left(c^2 + v^2(\gamma_1)^2(\gamma_0)^2\right)(\gamma_0)^2 \\ &= c^2\left(\frac{dt}{d\lambda}\right)^2\left(1 - (v/c)^2\right)(\gamma_0)^2\end{aligned}$$

Writing in a signature independent fashion (since both $(\gamma_0)^2 = 1$, and $(\gamma_0)^2 = -1$ can be picked), the absolute event arc length is

$$\begin{aligned}\left(\frac{ds}{d\lambda}\right)^2 &= \left|\frac{dx}{d\lambda}\right|^2 \\ &= (\gamma_0)^2\left(\frac{dx}{d\lambda}\right)^2\end{aligned}$$

$$\delta s = c\int_{t_1}^{t_2}\sqrt{1-(v/c)^2}dt$$

Writing $\delta s = c\delta\tau$, we have the proper time difference for fixed velocity motion

$$\delta\tau = \sqrt{1-(v/c)^2}\delta t$$

More generally, still employing the orthonormal Minkowski basis of SR for a particular event

$$\begin{aligned}x &= x^\mu\gamma_\mu \\ x^\mu &= x^\mu(\lambda)\end{aligned}$$

Taking derivatives along the path, we have

$$\begin{aligned}\frac{dx}{d\lambda} &= \frac{dx^\mu}{d\lambda}\gamma_\mu \\ \left(\frac{dx}{d\lambda}\right)^2 = \frac{dx}{d\lambda}\cdot\frac{dx}{d\lambda} &= \gamma_\mu\cdot\gamma_\nu\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \\ &= (\gamma_\mu)^2\left(\frac{dx^\mu}{d\lambda}\right)^2.\end{aligned}$$

This yields the arc length along an arbitrary space time path

$$\delta s = \int_{\lambda_1}^{\lambda_2} \sqrt{(\gamma_0)^2 (\gamma_\mu)^2 \left(\frac{dx^\mu}{d\lambda}\right)^2} d\lambda.$$

As in the constant velocity case, the proper time is a velocity scaled event arc length, and for any path parameterization one has

$$\delta \tau = \frac{1}{c} \int_{\lambda_1}^{\lambda_2} \sqrt{(\gamma_0)^2 (\gamma_\mu)^2 \left(\frac{dx^\mu}{d\lambda}\right)^2} d\lambda. \tag{5}$$

How is it reasonable to call this proper time, when it is really just represents 'four-vector-arc-length'? The rational for this is that when the particle is at rest ($dx^i/dt = 0$) the proper time then becomes

$$\begin{aligned}
\delta \tau &= \frac{1}{c} \int_{\lambda_1}^{\lambda_2} \sqrt{(\gamma_0)^4 \left(\frac{dx^0}{dt}\right)^2} dt \\
&= \frac{1}{c} \int_{\lambda_1}^{\lambda_2} c \frac{dt}{dt} dt \\
&= \delta t
\end{aligned}$$

The use of an orthonormal basis makes the metric tensor very simple. It has a diagonal form and is almost identity

$$\begin{aligned}
g_{\mu\nu} &= \gamma_\mu \cdot \gamma_\nu \\
&= \gamma_\mu^2 \delta_{\mu\nu} \\
&= (\gamma_0)^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}
\end{aligned}$$

Proper time in terms of this metric tensor is thus

$$\delta \tau = \frac{1}{c} \int_{\lambda_1}^{\lambda_2} \sqrt{(\gamma_0)^2 g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda. \tag{6}$$

3.2 A more general metric tensor in SR.

A more interesting metric tensor than the almost diagonal form associated with our orthonormal frame can be had by employing a more general basis f^μ. Write

$$x = f^\mu e_\mu = x^\mu \gamma_\mu$$

The only restriction on the set of vectors e_μ is that they span the four-space. That is sufficient to guarentee that a reciprocal frame can be calculated at any point

$$e^\nu \cdot e_\mu = \delta^\nu{}_\mu.$$

As in euclidian space the reciprocal frame can then be used to calculate the coordinates of any given point

$$f^\mu = x \cdot e^\mu.$$

Translation between this frame and the principle basis takes the form of a linear transformation

$$e_\mu = a^{\mu\alpha}\gamma_\alpha$$

Given both sets of vectors this change of basis function a can be calculated by taking dot products

$$\begin{aligned} e_\mu \cdot \gamma^\nu &= a^{\mu\alpha}\gamma_\alpha \cdot \gamma^\nu \\ &= a^{\mu\alpha}\delta_\alpha{}^\nu \\ &= a^{\mu\nu} \end{aligned}$$

$$\begin{aligned} e_\mu &= \left(e_\mu \cdot \gamma^\nu\right)\gamma_\nu && (7) \\ &= \left(e_\mu \cdot \gamma_\nu\right)\gamma^\nu && (8) \\ & && (9) \end{aligned}$$

This is enough to calculate difference in position (provided f^μ is not a function of position).

$$\begin{aligned} \left(\frac{dx}{d\lambda}\right)^2 &= e_\mu \cdot e_\nu \frac{df^\mu}{d\lambda}\frac{df^\nu}{d\lambda} \\ &= \left(e_\mu \cdot \gamma^\alpha\right)\left(e_\nu \cdot \gamma_\beta\right)\gamma_\alpha \cdot \gamma^\beta \frac{df^\mu}{d\lambda}\frac{df^\nu}{d\lambda} \\ &= \left(e_\mu \cdot \gamma^\alpha\right)\left(e_\nu \cdot \gamma_\alpha\right)\frac{df^\mu}{d\lambda}\frac{df^\nu}{d\lambda} \end{aligned}$$

This gives us the proper time with respect to these more general coordinates and their associated basis vectors

$$\delta\tau = \frac{1}{c}\int_{\lambda_1}^{\lambda_2} \sqrt{(\gamma_0)^2 \left(e_\mu \cdot \gamma^\alpha\right) \left(e_\nu \cdot \gamma_\alpha\right) \frac{df^\mu}{d\lambda}\frac{df^\nu}{d\lambda}} d\lambda. \tag{10}$$

For this mess of dot products, introduction of a tensor

$$g_{\mu\nu} = (\gamma_0)^2 \left(e_\mu \cdot \gamma^\alpha\right) \left(e_\nu \cdot \gamma_\alpha\right) \tag{11}$$

allows for the expressing the proper time for these general coordinates in a more compact form

$$\delta\tau = \frac{1}{c}\int_{\lambda_1}^{\lambda_2} \sqrt{g_{\mu\nu}\frac{df^\mu}{d\lambda}\frac{df^\nu}{d\lambda}} d\lambda. \tag{12}$$

Both of the Rindler and the Schwartzchild metrics as stated below have the general form of this structure of equation 12 (this is in fact a more general form since both those metrics are diagonal).

3.3 Position variation of SR frame vectors.

This is all assuming that the spacetime frame is not a function of position or time, or the variation of the frame vectors is so small that it can be neglected.

In the more general case when the derivatives of the frame vectors are significant those will have to be considered to calculate distance.

Lets see what the SR metric looks like in this case.

Again write

$$x = f^\mu e_\mu$$

Allowing both $f^\mu = f^\mu(\lambda)$ and $e_\mu = e_\mu(f^\nu)$ to vary with position a differential change in position with respect to parameter λ is thus

$$\begin{aligned}\frac{dx}{d\lambda} &= \frac{df^\mu}{d\lambda}e_\mu + f^\nu\frac{\partial e_\nu}{\partial f^\mu}\frac{df^\mu}{d\lambda} \\ &= \frac{df^\mu}{d\lambda}\left(e_\mu + f^\nu\frac{\partial e_\nu}{\partial f^\mu}\right)\end{aligned}$$

Introducing shorthand $\partial_\mu \equiv \frac{\partial}{\partial f^\mu}$, and $\dot{f}^\mu = \frac{df^\mu}{d\lambda}$, the squared magnitude is

$$\begin{aligned}\left(\frac{dx}{d\lambda}\right)^2 &= \left(e_\mu + f^\alpha\partial_\mu e_\alpha\right)\cdot\left(e_\nu + f^\beta\partial_\nu e_\beta\right)\dot{f}^\mu\dot{f}^\nu \\ &= \left(e_\mu\cdot e_\nu + f^\beta e_\mu\cdot(\partial_\nu e_\beta) + f^\alpha(\partial_\mu e_\alpha)\cdot e_\nu + f^\alpha f^\beta(\partial_\mu e_\alpha)\cdot(\partial_\nu e_\beta)\right)\dot{f}^\mu\dot{f}^\nu\end{aligned}$$

The dot products term here is symmetric, so one can write

$$g_{\mu\nu}(f^\sigma) = e_\mu \cdot e_\nu + 2 f^\beta e_\mu \cdot (\partial_\nu e_\beta) + f^\alpha f^\beta (\partial_\mu e_\alpha) \cdot (\partial_\nu e_\beta)$$

and express the incremental arc length (ignoring potential metric signature sign adjustment) along the curve as

$$\left(\frac{dx}{d\lambda}\right)^2 (d\lambda)^2 = g_{\mu\nu}(f^\sigma) \dot{f}^\mu \dot{f}^\nu \tag{13}$$

This is the completely general kinetic form for the Lagrangian of equation 1. The variation results for this equation that are summarized in 2 are therefore also equations of motion for special relativity when we allow for expressing the four vector in generalized coordinates. They happen to also be how the equations of motion of general relativity are also expressed. This shows that there is likely a geometric special relativistic interpretation of any general relativity metric. The Rindler metric calculation below will illustrate this well due to its simplicity.

4 Some GR calculations.

4.1 Schwartzchild Metric.

After too much thought about the geometrical origins of more general metrics in relativity, lets take one of them and run with it, working through the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial q} = \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)$$

for what Lut calls the Schwartzchild metric.

$$-c^2 (d\tau)^2 = -c^2 a(r)(dt)^2 + b(r)(dr)^2 + r^2 (d\theta)^2 \tag{14}$$

$$a(r) = 1 - \kappa/r \tag{15}$$

$$b(r) = \frac{1}{a(r)} = \frac{r}{r-k} \tag{16}$$

Form the Lagrangian

$$\mathcal{L} = -c^2 = -c^2 a\dot{t}^2 + b\dot{r}^2 + r^2\dot{\theta}^2 \tag{17}$$

Some intermediate calculations will be useful

$$\frac{\partial a}{\partial r} = \kappa / r^2$$
$$\frac{\partial b}{\partial r} = -\kappa / (r-k)^2$$
$$\dot{a} = \frac{\kappa \dot{r}}{r^2}$$
$$\dot{b} = \frac{-\kappa \dot{r}}{(r-k)^2}$$

First calcuate the EOMs for the cyclic coordinates

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right)$$
$$0 = (2r^2\dot{\theta})'$$

$$\frac{\partial \mathcal{L}}{\partial t} = \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}}\right)$$
$$0 = (-2c^2 a\dot{t})'$$

Introducing two arbitrary integration constants we have

$$\dot{\theta} = \frac{A}{r^2} \tag{18}$$
$$\dot{t} = \frac{T}{a} \tag{19}$$

The equations for the non-cyclic coordinate r is

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right)$$
$$-c^2\frac{\partial a}{\partial r}\dot{t}^2 + \frac{\partial b}{\partial r}\dot{r}^2 + 2r\dot{\theta}^2 = (2b\dot{r})' = 2\dot{b}\dot{r} + 2b\ddot{r}$$
$$-c^2\frac{\kappa}{r^2}\dot{t}^2 - \frac{\kappa}{(r-k)^2}\dot{r}^2 + 2r\dot{\theta}^2 = -2\frac{\kappa\dot{r}^2}{(r-k)^2} + 2b\ddot{r}$$
$$-\frac{1}{2}c^2\frac{a\kappa}{r^2}\dot{t}^2 - \frac{1}{2}\frac{a\kappa}{(r-k)^2}\dot{r}^2 + ar\dot{\theta}^2 + \frac{a\kappa\dot{r}^2}{(r-k)^2} = \ddot{r}$$
$$-\frac{1}{2}c^2\frac{\kappa T^2}{ar^2} + \frac{1}{2}\frac{a\kappa}{(r-k)^2}\dot{r}^2 + a\frac{A^2}{r^3} =$$

From the Lagrangian itself 17, and the integrated cyclic equations 18 one can eliminate the $\dot{r}^2$ term above

$$
\begin{aligned}
b\dot{r}^2 &= -c^2 + c^2 a\dot{t}^2 - r^2\dot{\theta}^2 \\
\dot{r}^2 &= -ac^2 + c^2 a^2 \dot{t}^2 - ar^2\dot{\theta}^2 \\
&= c^2(T^2 - a) - a\frac{A^2}{r^2}
\end{aligned}
$$

$$
\begin{aligned}
\ddot{r} &= -\frac{1}{2}c^2\frac{\kappa T^2}{ar^2} + \frac{1}{2}\frac{a\kappa}{(r-\kappa)^2}\left(c^2(T^2 - a) - a\frac{A^2}{r^2}\right) + a\frac{A^2}{r^3} \\
&= \underbrace{-\frac{1}{2}c^2\frac{\kappa T^2}{(r-\kappa)r} + \frac{1}{2}c^2\frac{\kappa T^2}{(r-\kappa)r}}_{=0} - \frac{1}{2}\frac{\kappa c^2}{r^2} - \frac{1}{2}\frac{\kappa A^2}{r^4} + \frac{(r-\kappa)A^2}{r^4}
\end{aligned}
$$

A final collection of terms yields

$$
\ddot{r} = -\frac{1}{2}\frac{\kappa c^2}{r^2} + \frac{A^2}{r^3} - \frac{3\kappa A^2}{2r^4} \tag{20}
$$

Now, the interesting thing here is that the metric itself can be considered a source of gravitational acceleration. Let $\kappa c^2/2 = GM$, and form the proper acceleration

$$
\ddot{r} = -\frac{GM}{r^2} + \frac{A^2}{r^3} - \frac{3GMA^2}{c^2 r^4}
$$

Note the similarity now to Newtonian gravity.

Lets also eliminate the arbitrary integration constant, using the angular velocity at a specific reference point $A = \dot{\theta}r^2 = \Omega_0(r_0)^2$

$$
\ddot{r} = -\frac{GM}{r^2} + \frac{(\Omega_0)^2(r_0)^4}{r^3} - \frac{1}{c^2}3GM(\Omega_0)^2\left(\frac{r_0}{r}\right)^4 \tag{21}
$$

This calculation is missing physics content. A good comparison to radial EOM in Newtonian physics ought to be done to see which of these terms can also be found in a non-relativistic treatment. Lut also suggests that equation 21 can probably be applied to calculate planar precession, which would be a cool application.

Blatently missing here is an understanding of how the metric relates to the mass distribution (ie: this is apparently for a spherical mass distribution). A treatment that seems quite readable is the following translation of Schwartzchild's original paper [Schwarzschild(1916)].

Question: can the Schwartzchild metric be formulated in terms of a position dependent variable basis as the Rindler metric can?

4.2 Rindler Metric.

The Rindler metric has only $g_{00} = g_{00}(x^1)$ different from unity.

Corresponding to a generalized basis

$$
\begin{aligned}
e_0 &= \sqrt{g_{00}}\gamma_0 \\
e_i &= \gamma_i
\end{aligned}
$$

Without specifying this function specifically, lets try the calculation. Writing $f(x) = g_{00}(x^1)$, the arc length and corresponding Lagrangian is

$$
\begin{aligned}
-c^2(d\tau)^2 &= -c^2(dt^0)^2 f(x) + dx^2 \\
\mathcal{L} = -c^2 &= -c^2\dot{t}^2 f(x) + \dot{x}^2
\end{aligned}
$$

The generalized time coordinate is cyclic:

$$
\begin{aligned}
\left(-2f(x)c^2\dot{t}\right)' &= 0 \\
-2f(x)c^2\dot{t} &= -2c^2\kappa \\
\dot{t} &= \frac{\kappa}{f(x)}.
\end{aligned}
$$

For the x coordinate we have

$$
\begin{aligned}
2\ddot{x} &= -f'(x)c^2\dot{t}^2 \\
\ddot{x} &= -\frac{1}{2}f'(x)c^2\dot{t}^2 \\
&= -\frac{f'(x)c^2\kappa^2}{f^2}
\end{aligned}
$$

Which is

$$
\frac{d^2x}{d\tau^2} = c^2\kappa^2\frac{d}{dx}\left(\frac{1}{2f(x)}\right).
$$

For $f(x) = a + bx$, this is

$$\frac{d^2x}{d\tau^2} = -c^2\kappa^2\frac{b}{2(a+bx)^2}$$

For $f(x) = (a+bx)^2$, what I believe Lut used, this is

$$\frac{d^2x}{d\tau^2} = -c^2\kappa^2\frac{b}{(a+bx)^3}$$

For $f(x) = (a+bx)^{1/2}$, this is

$$\frac{d^2x}{d\tau^2} = \frac{1}{4}c^2\kappa^2 b(a+bx)^{-3/2}$$

FIXME: I can't reproduce the $\ddot{x} \propto 1/\sqrt{f}$ result that Lut had, nor what I initially calculated on paper. My initial paper calculation looked wrong later when typing up, since I'd messed up polynomial powers (probably since I was expecting a specific answer). The above was recalculated without specifying f(x) upfront to make it harder to mess up the powers.

Think that I saw in the wiki page on the Rindler metric that there were both g_{00} and g_{11} terms (with square roots) in both. Try with that, and also do the calculations to see that those match the Γ gravitional field equations as specified in [Schwarzschild(1916)].

As far as interpretation goes... I think this can be interpretted in the geometrical fashion that intuition was telling me was there. From a strictly SR point of view, if you calculate the equations of motion for a particle in a frame where the time basis vector increases with position, a particle at rest in that frame is observed to be accelerating from an external (w/ constant basis vectors) frame. This sort of general coordinate system variation can't neccessarily be interpretted as an Einstein gravitational field since he has constraints on the allowed metrics. Working through some examples of that field calculation with various metrics should be helpful to get a feel for things.

FIXME: firm up this intpretive statement with the math to make it more meaningful, and consider physical examples of motion in the corresponding systems.

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