Gaussian Surface invariance for radial field.

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1 Flux independence of surface.

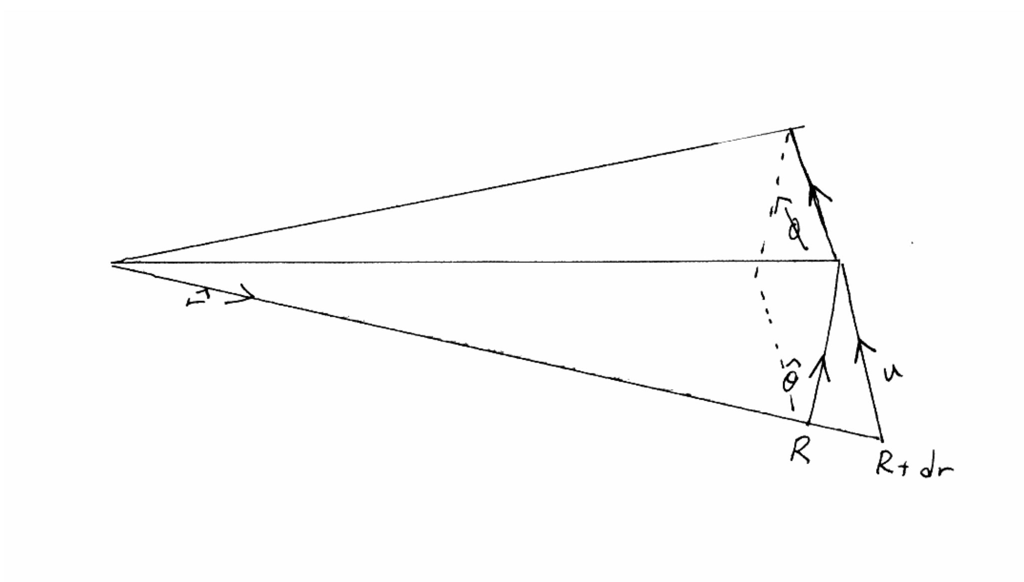

Figure 1: fig:Flux through tilted spherical surface element.

In [Purcell(1963)], section 1.10 is a demonstration that the flux through any closed surface is the same as that through a sphere.

A similar demonstration of the same is possible using a spherical polar basis $\{\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$ with an element of surface area that is tilted slightly as illustrated in figure 1.

The tangential surface on the sphere at radius r will have bivector

$$d\mathbf{A}_r = r^2 d\theta d\phi \hat{\boldsymbol{\theta}} \hat{\boldsymbol{\phi}} \tag{1}$$

where $d\theta$, and $d\phi$ are the subtended angles (should have put them in the figure).

Now, as in the figure we want to compute the bivector for the tilted surface at radius R. The vector $\mathbf{u}$ in the figure is required. This is $\hat{\mathbf{r}}R + Rd\theta\hat{\boldsymbol{\theta}} - \hat{\mathbf{r}}(R + dr)$, so the bivector for that area element is

$$\left(R\hat{\mathbf{r}} + Rd\theta\hat{\boldsymbol{\theta}} - (R+dr)\hat{\mathbf{r}}\right) \wedge Rd\theta\hat{\boldsymbol{\phi}} = \left(Rd\theta\hat{\boldsymbol{\theta}} - dr\hat{\mathbf{r}}\right) \wedge Rd\phi\hat{\boldsymbol{\phi}}$$

For

$$d\mathbf{A}_R = R^2 d\theta d\phi \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\phi}} - Rdrd\phi\hat{\mathbf{r}}\hat{\boldsymbol{\phi}} \tag{2}$$

Now normal area elements can be calculated by multiplication with a $\mathbb{R}^3$ pseudoscalar such as $I = \hat{\mathbf{r}}\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\phi}}$.

$$\begin{aligned}\hat{\mathbf{n}}_r|d\mathbf{A}_r| &= r^2 d\theta d\phi \hat{\mathbf{r}}\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\phi}}\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\phi}} \\ &= -r^2 d\theta d\phi \hat{\mathbf{r}}\end{aligned}$$

And

$$\begin{aligned}\hat{\mathbf{n}}_R|d\mathbf{A}_R| &= \hat{\mathbf{r}}\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\phi}}\left(R^2 d\theta d\phi \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\phi}} - Rdrd\phi\hat{\mathbf{r}}\hat{\boldsymbol{\phi}}\right) \\ &= -R^2 d\theta d\phi \hat{\mathbf{r}} - Rdrd\phi\hat{\boldsymbol{\theta}}\end{aligned}$$

Calculating $\mathbf{E} \cdot \hat{\mathbf{n}} dA$ for the spherical surface element at radius r we have

$$\begin{aligned}\mathbf{E}(r) \cdot \hat{\mathbf{n}}_r|d\mathbf{A}_r| &= \frac{1}{4\pi\epsilon_0 r^2} q\hat{\mathbf{r}} \cdot (-r^2 d\theta d\phi \hat{\mathbf{r}}) \\ &= \frac{-d\theta d\phi q}{4\pi\epsilon_0}\end{aligned}$$

and for the tilted surface at R

$$\begin{aligned}\mathbf{E}(R) \cdot \hat{\mathbf{n}}_R|d\mathbf{A}_R| &= \frac{q}{4\pi\epsilon_0 R^2}\hat{\mathbf{r}} \cdot \left(-R^2 d\theta d\phi \hat{\mathbf{r}} - Rdrd\phi\hat{\boldsymbol{\theta}}\right) \\ &= \frac{-d\theta d\phi q}{4\pi\epsilon_0}\end{aligned}$$

The $\hat{\boldsymbol{\theta}}$ component of the surface normal has no contribution to the flux since it is perpendicular to the outwards ($\hat{\mathbf{r}}$ facing) field. Here the particular normal to the surface happened to be inwards facing due to choice of the pseudoscalar, but because the normals chosen in each case had the same orientation this doesn't make a difference to the equivalence result.

1.1 Suggests dual form of Gauss's law can be natural.

The fact that the bivector area elements work well to describe the surface can also be used to write Gauss's law in an alternate form. Let $\hat{\mathbf{n}} dA = -I d\mathbf{A}$

$$\begin{aligned}
\mathbf{E} \cdot \hat{\mathbf{n}} dA &= -\mathbf{E} \cdot (I d\mathbf{A}) \\
&= \frac{-1}{2} (\mathbf{E} I d\mathbf{A} + I d\mathbf{A} \mathbf{E}) \\
&= \frac{-I}{2} (\mathbf{E} d\mathbf{A} + d\mathbf{A} \mathbf{E}) \\
&= -I (\mathbf{E} \wedge d\mathbf{A})
\end{aligned}$$

So for

$$\int \mathbf{E} \cdot \hat{\mathbf{n}} dA = \int \frac{\rho}{\epsilon_0} dV$$

with $d\mathbf{V} = I dV$, we have Gauss's law in dual form:

$$\int \mathbf{E} \wedge d\mathbf{A} = \int \frac{\rho}{\epsilon_0} d\mathbf{V}$$

Writing Gauss's law in this form it becomes almost obvious that we can deform the surface without changing the flux, since all the non-tangential surface elements will have an $\hat{\mathbf{r}}$ factor and thus produce a zero once wedged with the radial field.

References

[Purcell(1963)] E.M. Purcell. *Electricity and magnetism (Berkeley physics course, v. 2)*. McGraw-hill, 1963.