Some rapidity angle notes.

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Dec 18, 2008. Last Revision: $Date : 2008/12/1904 : 36 : 09$

1 Motivation.

Lut writes, "setting up a little calculation, I'm writing a 4-velocity as"

$$(\gamma, \gamma\beta_x, \gamma\beta_y, \gamma\beta_z),$$

which has length -1 if $gamma^{-2} = 1 - (\beta_x)^2 + (\beta_y)^2 + (\beta_z)^2$.

Can you write this in terms of the 3 rapidities a_1, a_2, a_3 ?

I wasn't able to answer this right away so it is worth an examination of rapidity angles to ensure that I understand the ideas.

2 Stuff.

Putting back in the c factors, and switching to the $+---$ signature I'm used to, the position vector is

$$x = x^\mu \gamma_\mu = ct\gamma_0 + x^i \gamma_i,$$

for which the corresponding proper velocity is

$$v = \frac{dx}{d\tau} = c\frac{dt}{d\tau}\gamma_0 + \frac{dx^i}{dt}\frac{dt}{d\tau}\gamma_i$$

Writing $\gamma = dt/d\tau$, and squaring the proper velocity we have

$$\begin{aligned}
\frac{v^2}{c^2} &= 1 \\
&= \gamma^2 \left(\gamma_0 + \frac{1}{c}\frac{dx^i}{dt}\gamma_i\right)^2 \\
&= \gamma^2 \left(1 - \sum_i \frac{1}{c^2}\left(\frac{dx^i}{dt}\right)^2\right)
\end{aligned}$$

So we have

$$\gamma = \frac{1}{\sqrt{1 - \sum_i \frac{1}{c^2} \left(\frac{dx^i}{dt}\right)^2}}$$

Observe that γ ranges from 1 to infinity, and can thus be described by the $[0, \infty]$ range of the hyperbolic cosine function. With the relative velocity $\mathbf{v} = \sum_i (dx^i/dt)\sigma_i$, this is

$$\begin{aligned}
\frac{dt}{d\tau} &= \gamma \\
&= \cosh \alpha \\
&= \frac{1}{\sqrt{1 - (\mathbf{v}/c)^2}}
\end{aligned}$$

In terms of the hyperbolic cosine for γ our proper velocity then becomes

$$v/c = \cosh \alpha \left(1 + \frac{\mathbf{v}}{c}\right) \gamma_0$$

Taking the hint from the Lorentz transform where we have both sinh and cosh factors can one write

$$\gamma \frac{\mathbf{v}}{c} = \sinh \alpha$$

This gives

$$\frac{\mathbf{v}}{c} = \tanh \alpha$$

so we need α to be a spacetime relative vector. With cosh being an even function $\cosh \alpha = \cosh |\alpha|$, so this is still a scalar as desired. Inverting the relationship for α we have

$$\alpha = \tanh^{-1}(\mathbf{v}/c) = \hat{\mathbf{v}} \tanh^{-1}(|\mathbf{v}/c|)$$

The unit vector $\hat{\mathbf{v}}$ can be factored out of the inverse hyperbolic tangent function since it is odd (consider the Taylor series expansion of $\tanh^{-1}$ to see why one can do this).

Finally, we have by dotting with the spatial basis vectors σ_i three quantities in terms of spacetime vector rapidity angle

$$\alpha_i = (\hat{\mathbf{v}} \cdot \sigma_i) \tanh^{-1}(|\mathbf{v}/c|).$$

The $\hat{\mathbf{v}} \cdot \sigma_i$ parts are direction cosines, so the three rapidities Lut was asking about all appear to be weighted direction cosines.