Ad-hoc motivation of some QM wave equations.

Peeter Joot peeter.joot@gmail.com

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1 Motivation.

1.1 Non-Relativistic case.

A common (cf: wikipedia and [French and Taylor(1998)]) introductory motivation for the non-relativistic Schrödinger's equation appears to follow the following lines. Assume that we desire a plane wave equation of the following form

$$\psi = \exp(i(\mathbf{k} \cdot \mathbf{x} - \omega t))$$

plus a requirement that we have a total energy that can be expressed in terms of kinetic plus potential

$$E = \frac{\mathbf{p}^2}{2m} + V \tag{1}$$

We also have the Einstein relationship

$$E = h\nu = \frac{h}{2\pi} 2\pi\nu = \hbar\omega \tag{2}$$

and the Debroglie Hypothosis for the magnitude of the momentum of a particle

$$p = \frac{h}{\lambda}.$$

In terms of wave number $2\pi k = \frac{1}{\lambda}$ this last is

$$p = \frac{h}{2\pi} k = \hbar k.$$

or in three dimensions

$$\mathbf{p} = \hbar \mathbf{k}.$$

Taking derivatives of the postulated wave function one has

$$\frac{\partial \psi}{\partial t} = -i\omega\psi \tag{3}$$

and

$$\boldsymbol{\nabla}^2 \psi = i^2 \sum_j k_j^2 \psi = -\mathbf{k}^2 \psi \tag{4}$$

From the energy relationship, if one requires that

$$E\psi = \left(\frac{\mathbf{p}^2}{2m} + V \right) \psi \tag{5}$$

$$\hbar\omega\psi = \left(\hbar^2 \frac{\mathbf{k}^2}{2m} + V \right) \psi \tag{6}$$

$$\tag{7}$$

and then substuiting the derivatives from equations 3 and 4 we have

$$i\hbar \frac{\partial \psi}{\partial t} = \left(-\frac{\hbar^2}{2m} \boldsymbol{\nabla}^2 + V \right) \psi$$

1.2 Relativistic case.

The relativistic force free Schrödinger's equation is motivated by [Srednicki(2007)] replacing the Hamiltonian operator $H = \mathbf{P}^2/2m$ with

$$H = \sqrt{m^2c^4 + \mathbf{P}^2c^2} \approx mc^2 + \mathbf{P}^2/2m$$

for

$$i\hbar\frac{\partial\psi}{\partial t} = \sqrt{-\hbar^2 c^2 \boldsymbol{\nabla}^2 + m^2 c^4}\psi$$

then squaring the operators on both sides, removing the root:

$$\begin{aligned}
-\hbar^2 \frac{\partial^2\psi}{(\partial x^0)^2} &= \left(-\hbar^2 \boldsymbol{\nabla}^2 + m^2 c^2\right)\psi \\
-\hbar^2 \frac{\partial^2\psi}{(\partial x^0)^2} + \hbar^2 \boldsymbol{\nabla}^2 &= m^2 c^2 \psi
\end{aligned}$$

Which is called the Klein-Gordon equation:

$$\left(\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}mc^2\right)\psi = 0$$

The Noether's theorem wikipedia article, and Klein-Gordon pages give the action as (removing use of natural units and changing to a $+---$ metric) gives:

$$\begin{aligned}
\mathcal{L} &= -\eta^{\mu\nu}\partial_\mu\psi\partial_\nu\psi^* + \frac{m^2c^2}{\hbar^2}\psi\psi^* \\
&= -\partial^\nu\psi\partial_\nu\psi^* + \frac{m^2c^2}{\hbar^2}\psi\psi^*
\end{aligned}$$

This first term is a squared spacetime gradient

$$\begin{aligned}
(\nabla\psi)\cdot(\nabla\psi^*) &= (\gamma^\mu\partial_\mu\psi)\cdot(\gamma_\nu\partial^\nu\psi^*) \\
&= \delta^\mu{}_\nu\partial_\mu\psi\partial^\nu\psi^* \\
&= \partial_\mu\psi\partial^\mu\psi^*
\end{aligned}$$

so we have

$$\begin{aligned}
\mathcal{L} &= -\eta^{\mu\nu}\partial_\mu\psi\partial_\nu\psi^* + \frac{m^2c^2}{\hbar^2}\psi\psi^* \\
&= -(\nabla\psi)\cdot(\nabla\psi^*) + \frac{m^2c^2}{\hbar^2}\psi\psi^*
\end{aligned}$$

So, here we expect to get a spacetime Laplacian, like the Maxwell potential equation, and one does:

$$-\nabla^2 \psi = \frac{m^2 c^2}{\hbar^2} \psi$$

Now, the Srednicki text indicates that Dirac linearized this equation, to get back something that was first order in the time derivative.

I don't yet follow that description, but see that a linearization is possible by taking roots of the operators above, undoing the somewhat fishy seeming squaring done to arrive at the Klein-Gordon equation in the first place

$$i\hbar \nabla \psi = \pm mc\psi \tag{8}$$

Is this equivalent to Dirac's formulation? Comparision to , where the dirac equation is given in covariant form

$$i\hbar \gamma^\mu \partial_\mu \psi - mc\psi = 0$$

which is the positive variant of 8.

What is the Lagrangian that is associated with this? The most probable interpretation of i here is the Minkowski pseudoscalar, as opposed to a unit bivector of spacetime vectors or some other geometrical object with -1 square.

References

[French and Taylor(1998)] A.P. French and E.F. Taylor. *An Introduction to Quantum Physics*. CRC Press, 1998.

[Srednicki(2007)] M.A. Srednicki. *Quantum Field Theory*. Cambridge University Press, 2007.