Expressing wave equation exponential solutions using four vectors.

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1 Mechanical Wave equation Solutions.

For the unforced wave equation in 3D one wants solutions to

$$\left(\frac{1}{\mathbf{v}^2}\partial_{tt} - \sum_{j=1}^{3}\partial_{jj}\right)\phi = 0 \tag{1}$$

For the single spatial variable case one can verify that $\phi = f(\mathbf{x} \pm |\mathbf{v}|t)$ is a solution for any function f. In particular $\phi = \exp(i(\pm|\mathbf{v}|t + x))$ is a solution. Similarily $\phi = \exp(i(\pm|\mathbf{v}|t + \hat{\mathbf{k}} \cdot \mathbf{x}))$ is a solution in the 3D case.

Can the relativistic four vector notation be used to put this in a more symmetric form with respect to time and position? For the four vector

$$x = x^\mu \gamma_\mu$$

Lets try the following as a possible solution to 1

$$\phi = \exp(ik \cdot x)$$

verifying that this can be a solution, and determining the constraints required on the four vector k.

Observe that

$$x \cdot k = x^\mu k_\mu$$

so

$$\begin{aligned}
\phi_\mu &= ik_\mu \\
\phi_{\mu\mu} &= (ik_\mu)^2\phi = -(k_\mu)^2\phi
\end{aligned}$$

Since $\partial_t = c\partial_0$, we have $\phi_t t = c^2\phi_{00}$, and

$$\left(\frac{1}{\mathbf{v}^2}\partial_{tt} - \sum_{j=1}^{3}\partial_{jj}\right)\phi = \left(-\frac{1}{\mathbf{v}^2}c^2{k_0}^2 - \sum_{j=1}^{3} -(k_j)^2\right)\phi$$

For equality with zero, and $\boldsymbol{\beta} = \mathbf{v}/c$, we require

$$\boldsymbol{\beta}^2 = \frac{(k_0)^2}{\sum_j (k_j)^2}$$

Now want the components of $k = k_\mu \gamma^\mu$ in terms of k directly. First

$$k_0 = k \cdot \gamma_0$$

The spacetime relative vector for k is

$$\mathbf{k} = k \wedge \gamma_0 = \sum k_\mu \gamma^\mu \wedge \gamma_0 = (\gamma_1)^2 \sum_j k_j \sigma_j$$

$$\mathbf{k}^2 = (\pm 1)^2 \sum_j (k_j)^2$$

So the constraint on the four vector parameter k is

$$\begin{aligned}\boldsymbol{\beta}^2 &= \frac{(k_0)^2}{\sum_j (k_j)^2} \\ &= \frac{(k \cdot \gamma_0)^2}{(k \wedge \gamma_0)^2}\end{aligned}$$

It is interesting to compare this to the relative spacetime bivector for x

$$\begin{aligned}v &= \frac{dx}{d\tau} = c\frac{dt}{d\tau}\gamma_0 + \frac{dx^i}{d\tau}\gamma_i \\ v \cdot \gamma^0 &= \frac{dx}{d\tau} \cdot \gamma^0 = c\frac{dt}{d\tau} \\ v \wedge \gamma_0 &= \frac{dx}{d\tau} \wedge \gamma_0 \\ &= \frac{dx^i}{d\tau}\sigma_i \\ &= \frac{dx^i}{dt}\frac{dt}{d\tau}\sigma_i\end{aligned}$$

$$
\begin{aligned}
\mathbf{v}/c &= \frac{d(x^i \sigma_i)}{dt} \\
&= \frac{v \wedge \gamma_0}{v \cdot \gamma^0}
\end{aligned}
$$

So, for $\phi = \exp(ik \cdot x)$ to be a solution to the wave equation for a wave travelling with velocity $|\mathbf{v}|$, the constraint on k in terms of proper velocity v is

$$
\left| \frac{k \wedge \gamma_0}{k \cdot \gamma^0} \right|^{-1} = \left| \frac{v \wedge \gamma_0}{v \cdot \gamma^0} \right| \tag{2}
$$

So we see the relative spacetime vector of k has an inverse relationship with the relative spacetime velocity vector $v = dx/d\tau$.