Relativistic acceleration.

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Contents

1 Motivation.

Continuing on with reading of [Pauli(1981)], having clarified aspects of the four vector velocity in [Joot()], it is now time to move on to acceleration.

Do the chain rule calculations for the acceleration four vector equation given in equation (193).

2 Compute it.

Compute the spatial and timelike components of the acceleration

$$
\begin{aligned}
B^\mu &= \frac{d^2 x^\mu}{d\tau^2} \\
&= \frac{d}{d\tau}\left(\frac{dx^\mu}{d\tau}\right) \\
&= \frac{d}{d\tau}\left(\frac{dx^\mu}{dt}\frac{dt}{d\tau}\right) \\
&= \left(\frac{d}{d\tau}\frac{dx^\mu}{dt}\right)\frac{dt}{d\tau} + \frac{dx^\mu}{dt}\frac{d^2 t}{d\tau^2} \\
&= \frac{d^2 x^\mu}{dt^2}\left(\frac{dt}{d\tau}\right)^2 + \frac{dx^\mu}{dt}\frac{d^2 t}{d\tau^2}
\end{aligned}
$$

For $\mu \in \{1, 2, 3\}$, the d^2x^μ/dt^2 terms are the regular old spatial acceleration components. , and $dx^4/dt = c$. Writing $\mathbf{u}^2 = \sum_{k=1}^3 (dx^k/dt)^2$, and $\beta^2 = \mathbf{u}^2/c^2$, we have

$$
\begin{aligned}
B^k &= \frac{d^2x^k}{dt^2}\frac{1}{1-\beta^2} + \frac{dx^k}{dt}\frac{d^2t}{d\tau^2} \\
B^4 &= 0 + c\frac{d^2t}{d\tau^2}
\end{aligned}
$$

In both of these is the $d^2t/d\tau^2$ term. Let's expand that.

$$
\begin{aligned}
\frac{d^2t}{d\tau^2} &= \frac{d}{d\tau}\left(\frac{1}{\sqrt{1-\mathbf{u}^2/c^2}}\right) \\
&= \frac{-1}{c^2}\frac{(-1/2)}{(1-\mathbf{u}^2/c^2)^{3/2}}\frac{d\mathbf{u}^2}{d\tau} \\
&= \frac{1}{c^2}\frac{(1/2)}{(1-\mathbf{u}^2/c^2)^{3/2}}2\mathbf{u}\cdot\frac{d\mathbf{u}}{d\tau} \\
&= \frac{1}{c^2}\frac{1}{(1-\mathbf{u}^2/c^2)^{3/2}}\mathbf{u}\cdot\frac{d\mathbf{u}}{dt}\frac{dt}{d\tau} \\
&= \frac{1}{c^2}\frac{1}{(1-\mathbf{u}^2/c^2)^2}\mathbf{u}\cdot\frac{d\mathbf{u}}{dt}
\end{aligned}
$$

In vector form, with $\mathbf{a} = d\mathbf{u}/dt$, we now have the following

$$
\mathbf{B} = \mathbf{a}\frac{1}{1-\beta^2} + \mathbf{u}(\mathbf{u}\cdot\mathbf{a})\frac{1}{c^2}\frac{1}{(1-\beta^2)^2} \tag{1}
$$

$$
B^4 = \frac{1}{c}(\mathbf{u}\cdot\mathbf{a})\frac{1}{(1-\beta^2)^2} \tag{2}
$$

This reproduces the equation from the Pauli text (except for the imaginary factor i due to the Minkowski notation).

3 Approximate expansion.

This relativistic acceleration should match the Newtonian acceleration for small velocities. Lets expand it to verify and inspect the form. Taylor expansions of the γ factors is required.

$$
\begin{aligned}
\frac{1}{1-\beta^2} &= 1 + \frac{(-1)}{1!}(-\beta^2) + \frac{(-1)(-2)}{2!}(-\beta^2)^2 + \frac{(-1)(-2)(-3)}{3!}(-\beta^2)^3 + \cdots \\
&= 1 + \beta^2 + \beta^4 + \beta^6 + \cdots
\end{aligned}
$$

This is convergent since $\beta < 1$, and for non-relativistic rates the higher order terms die off very quickly.

For the γ^4 term we want

$$
\begin{aligned}
\frac{1}{(1-\beta^2)^2} &= 1 + \frac{(-2)}{1!}(-\beta^2) + \frac{(-2)(-3)}{2!}(-\beta^2)^2 + \frac{(-2)(-3)(-4)}{3!}(-\beta^2)^3 + \cdots \\
&= 1 + 2\beta^2 + 3\beta^4 + 4\beta^6 + \cdots
\end{aligned}
$$

Again, this is convergent. Substuition back into 1 we have for the spatial part

$$
\mathbf{B} = \mathbf{a}\left(1 + \beta^2 + \beta^4 + \beta^6 + \cdots\right) + \mathbf{u}(\mathbf{u}\cdot\mathbf{a})\frac{1}{c^2}\left(1 + 2\beta^2 + 3\beta^4 + 4\beta^6 + \cdots\right)
$$

Writing $\boldsymbol{\beta} = \mathbf{u}/c$, this is

$$
\mathbf{B} = \mathbf{a} + \boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{a}) + \boldsymbol{\beta}^2\left(\mathbf{a} + 2\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{a})\right) + \boldsymbol{\beta}^4\left(\mathbf{a} + 3\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{a})\right) + \cdots
$$

for small $|\boldsymbol{\beta}|$ we have the Newtonian acceleration. Another case that kills off terms is the circular motion condition $\boldsymbol{\beta}\cdot\mathbf{a} = 0$, for which we have just

$$
\mathbf{B} = \mathbf{a}\left(1 + \beta^2 + \beta^4 + \beta^6 + \cdots\right)
$$

So for circular motion the first order of magnitude correction to the acceleration is

$$
\mathbf{B} = \mathbf{a}(1 + \mathbf{u}^4/c^4)
$$

On the other hand for non-circular motion the more general first adjustment to the Newtonian acceleration is

$$
\begin{aligned}
\mathbf{B} &= \mathbf{a} + \frac{\mathbf{u}}{c}\left|\frac{\mathbf{u}}{c}\right||\mathbf{a}|\cos(\mathbf{u}, \mathbf{a}) \\
&= \mathbf{a} + \hat{\mathbf{u}}\left(\frac{\mathbf{u}}{c}\right)^2 |\mathbf{a}|\cos(\mathbf{u}, \mathbf{a}) \\
&= \hat{\mathbf{u}}|\mathbf{a}|\cos(\mathbf{u}, \mathbf{a})\left(1 + \left(\frac{\mathbf{u}}{c}\right)^2\right) + \hat{\mathbf{u}}\left(\hat{\mathbf{u}} \wedge \mathbf{a}\right)
\end{aligned}
$$

Or, putting back the explicit dot products

$$
\mathbf{B} = \hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \mathbf{a})\left(1 + \left(\frac{\mathbf{u}}{c}\right)^2\right) + \hat{\mathbf{u}}\left(\hat{\mathbf{u}} \wedge \mathbf{a}\right)
$$

We see here that we have a scale correction only in the direction of the projection of the accereration onto the direction of the velocity, and in the perpendicular direction to the acceleration the components go untouched.

References

[Joot()] Peeter Joot. Four vector velocity addition notes. "http://sites.google.com/site/peeterjoot/math2009/pauli_four_vector_v.pdf".

[Pauli(1981)] W. Pauli. *Theory of Relativity*. Dover Publications, 1981.