Biot Savart Derivation.

Peeter Joot   peeter.joot@gmail.com

April 18, 2009. Last Revision: *Date* : 2009/04/1804 : 30 : 57

Contents

1 Motivation.

Looked at my Biot-Savart derivation in [Joot()]. There I was playing with doing this without first dropping down to the familiar vector relations, and end up with an expression of the Biot Savart law in terms of the complete Faraday bivector. This is an excessive approach, albiet interesting (to me). Let's try this again in terms of just the magnetic field.

2 Do it.

2.1 Setup. Ampere-Maxwell equation for steady state.

The starting point can still be Maxwell's equation

$$\nabla F = J/\epsilon_0 c \tag{1}$$

and the approach taken will be the more usual consideration of a loop of steady-state (no-time variation) current.

In the steady state we have

$$\nabla = \gamma^0 \frac{1}{c}\partial_t + \gamma^k \partial_k = \gamma^k \partial_k$$

and in particular

$$\gamma_0 \nabla F = \gamma_0 \gamma^k \partial_k F$$
$$= \gamma_k \gamma_0 \partial_k F$$
$$= \sigma_k \partial_k F$$
$$= \boldsymbol{\nabla}(\mathbf{E} + ic\mathbf{B})$$

and for the RHS,

$$\gamma_0 J / \epsilon_0 c = \gamma_0 (c\rho \gamma_0 + J^k \gamma_k) / \epsilon_0 c$$
$$= (c\rho - J^k \sigma_k) / \epsilon_0 c$$
$$= (c\rho - \mathbf{j}) / \epsilon_0 c$$

So we have

$$\boldsymbol{\nabla}(\mathbf{E} + ic\mathbf{B}) = \frac{1}{\epsilon_0}\rho - \frac{\mathbf{j}}{\epsilon_0 c} \tag{2}$$

Selection of the (spatial) vector grades gives

$$ic(\boldsymbol{\nabla} \wedge \mathbf{B}) = -\frac{\mathbf{j}}{\epsilon_0 c}$$

or with $\mathbf{a} \wedge \mathbf{b} = i(\mathbf{a} \times \mathbf{b})$, and $\epsilon_0 \mu_0 c^2 = 1$, this is the familar Ampere-Maxwell equation when $\partial \mathbf{E} / \partial t = 0$.

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j} \tag{3}$$

2.2 Three vector potential solution.

With $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ (the trivector part of 2), we can write

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$$

For some vector potential $\mathbf{A}$. In particular, we have in 3,

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A})$$
$$= -i(\nabla \wedge (\nabla \times \mathbf{A}))$$
$$= -\frac{i}{2}(\nabla(\nabla \times \mathbf{A}) - (\nabla \times \mathbf{A})\nabla)$$
$$= \frac{i^2}{2}(\nabla(\nabla \wedge \mathbf{A}) - (\nabla \wedge \mathbf{A})\nabla)$$
$$= -\nabla \cdot (\nabla \wedge \mathbf{A})$$

Therefore the three vector potential equation for the magnetic field is

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{j} \tag{4}$$

2.3 Gauge freedom.

We have the freedom to set $\nabla \cdot \mathbf{A} = 0$, in 4. To see this suppose that the vector potential is expressed in terms of some other potential $\mathbf{A}'$ that does have zero divergence ($\nabla \cdot \mathbf{A}' = 0$) plus a (spatial) gradient

$$\mathbf{A} = \mathbf{A}' + \nabla \phi$$

Provided such a construction is possible, then we have

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \nabla(\nabla \cdot (\mathbf{A}' + \nabla \phi)) - \nabla^2(\mathbf{A}' + \nabla \phi)$$
$$= -\nabla^2 \mathbf{A}'$$

and can instead solve the simpler equivalent problem

$$\nabla^2 \mathbf{A}' = -\mu_0 \mathbf{j} \tag{5}$$

Addition of the gradient $\nabla \phi$ to A' will not change the magnetic field $\mathbf{B}$ since $\nabla \times (\nabla \phi) = 0$.

FIXME: what wasn't shown here is that it is possible to express any vector potential $\mathbf{A}$ in terms of a divergence free potential and a gradient. How would one show this?

2.4 Solution to the vector Poisson equation.

The solution (dropping primes) to the Poisson equation 5 is

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}}{r} dV$$

(See [Schwartz(1987)] for example.)

The magnetic field follows by taking the spatial curl

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$$

$$= \frac{\mu_0}{4\pi} \boldsymbol{\nabla} \times \int \frac{\mathbf{j}'}{|\mathbf{r} - \mathbf{r}'|} dV'$$

Pulling the curl into the integral and writing the gradient in terms of radial components

$$\boldsymbol{\nabla} = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \frac{\partial}{\partial |\mathbf{r} - \mathbf{r}'|}$$

we have

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \times \mathbf{j}' \frac{\partial}{\partial |\mathbf{r} - \mathbf{r}'|} \frac{1}{|\mathbf{r} - \mathbf{r}'|} dV'$$

$$= -\frac{\mu_0}{4\pi} \int \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \times \mathbf{j}' dV'$$

Finally with $\mathbf{j}' dV' = I\hat{\mathbf{j}}' dl'$, we have

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int dl' \hat{\mathbf{j}}' \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \tag{6}$$

References

[Joot()] Peeter Joot. Electrostatic field energy. "http://sites.google.com/site/peeterjoot/math/electric_field_energy.pdf".

[Schwartz(1987)] M. Schwartz. *Principles of Electrodynamics*. Dover Publications, 1987.