Energy momentum tensor relation to Lorentz force.

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1 Motivation.

In [Joot(b)] the energy momentum tensor was related to the Lorentz force in STA form. Work the same calculation strictly in tensor form, to develop more comfort with tensor manipulation. This should also serve as a translation aid to compare signs due to metric tensor differences in other reading.

1.1 Definitions.

The energy momentum "tensor", really a four vector, is defined in [Doran and Lasenby(2003)] as

$$T(a) = \frac{\epsilon_0}{2} F a \tilde{F} = -\frac{\epsilon_0}{2} F a F \tag{1}$$

We've seen that the divergence of the $T(\gamma^\mu)$ vectors generate the Lorentz force relations.

Let's expand this with respect to index lower basis vectors for use in the divergence calculation.

$$T(\gamma^\mu) = (T(\gamma^\mu) \cdot \gamma^\nu)\gamma_\nu \tag{2}$$

So we define

$$T^{\mu\nu} = T(\gamma^\mu) \cdot \gamma^\nu \tag{3}$$

and can write these four vectors in tensor form as

$$T(\gamma^\mu) = T^{\mu\nu}\gamma_\nu \tag{4}$$

1.2 Expanding out the tensor.

An expansion of $T^{\mu\nu}$ was done in [Joot(a)], but looking back that seems a peculiar way, using the four vector potential.

Let's try again in terms of $F^{\mu\nu}$ instead. Our field is

$$F = \frac{1}{2} F^{\mu\nu} \gamma_\mu \wedge \gamma_\nu \tag{5}$$

So our tensor components are

$$\begin{aligned}
T^{\mu\nu} &= T(\gamma^\mu) \cdot \gamma^\nu \\
&= -\frac{\epsilon_0}{8} F^{\lambda\sigma} F^{\alpha\beta} \left\langle (\gamma_\lambda \wedge \gamma_\sigma) \gamma^\mu (\gamma_\alpha \wedge \gamma_\beta) \gamma^\nu \right\rangle
\end{aligned}$$

Or

$$\begin{aligned}
-8 \frac{1}{\epsilon_0} T^{\mu\nu} &= F^{\lambda\sigma} F^{\alpha\beta} \left\langle (\gamma_\lambda {\delta_\sigma}^\mu - \gamma_\sigma {\delta_\lambda}^\mu)(\gamma_\alpha {\delta_\beta}^\nu - \gamma_\beta {\delta_\alpha}^\nu) \right\rangle \\
&\quad + F^{\lambda\sigma} F^{\alpha\beta} \left\langle (\gamma_\lambda \wedge \gamma_\sigma \wedge \gamma^\mu)(\gamma_\alpha \wedge \gamma_\beta \wedge \gamma^\nu) \right\rangle
\end{aligned}$$

Expanding only the first term to start with

$$\begin{aligned}
& F^{\lambda\sigma} F^{\alpha\beta} (\gamma_\lambda {\delta_\sigma}^\mu) \cdot (\gamma_\alpha {\delta_\beta}^\nu) + F^{\lambda\sigma} F^{\alpha\beta} (\gamma_\sigma {\delta_\lambda}^\mu) \cdot (\gamma_\beta {\delta_\alpha}^\nu) - F^{\lambda\sigma} F^{\alpha\beta} (\gamma_\lambda {\delta_\sigma}^\mu) \cdot (\gamma_\beta {\delta_\alpha}^\nu) - F^{\lambda\sigma} F^{\alpha\beta} (\gamma_\sigma {\delta_\lambda}^\mu) \cdot (\gamma_\alpha {\delta_\beta}^\nu) \\
&= F^{\lambda\mu} F^{\alpha\nu} \gamma_\lambda \cdot \gamma_\alpha + F^{\mu\sigma} F^{\nu\beta} \gamma_\sigma \cdot \gamma_\beta - F^{\lambda\mu} F^{\nu\beta} \gamma_\lambda \cdot \gamma_\beta - F^{\mu\sigma} F^{\alpha\nu} \gamma_\sigma \cdot \gamma_\alpha \\
&= \eta_{\alpha\beta} (F^{\lambda\mu} F^{\alpha\nu} \gamma_\lambda \cdot \gamma^\beta + F^{\mu\sigma} F^{\nu\alpha} \gamma_\sigma \cdot \gamma^\beta - F^{\lambda\mu} F^{\nu\alpha} \gamma_\lambda \cdot \gamma^\beta - F^{\mu\sigma} F^{\alpha\nu} \gamma_\sigma \cdot \gamma^\beta) \\
&= \eta_{\alpha\lambda} F^{\lambda\mu} F^{\alpha\nu} + \eta_{\alpha\sigma} F^{\mu\sigma} F^{\nu\alpha} - \eta_{\alpha\lambda} F^{\lambda\mu} F^{\nu\alpha} - \eta_{\alpha\sigma} F^{\mu\sigma} F^{\alpha\nu} \\
&= 2(\eta_{\alpha\lambda} F^{\lambda\mu} F^{\alpha\nu} + \eta_{\alpha\sigma} F^{\mu\sigma} F^{\nu\alpha}) \\
&= 4 \eta_{\alpha\beta} F^{\beta\mu} F^{\alpha\nu} \\
&= 4 F^{\beta\mu} {F_\beta}^\nu \\
&= 4 F^{\alpha\mu} {F_\alpha}^\nu
\end{aligned}$$

For the second term after a shuffle of indexes we have

$$F^{\lambda\sigma} F_{\alpha\beta} \eta^{\mu\mu'} \left\langle (\gamma_\lambda \wedge \gamma_\sigma \wedge \gamma_\mu)(\gamma^\alpha \wedge \gamma^\beta \wedge \gamma^\nu) \right\rangle$$

This dot product is reducable with the identity

$$(a \wedge b \wedge c) \cdot (d \wedge e \wedge f) = (((a \wedge b \wedge c) \cdot d) \cdot e) \cdot f$$

leaving a completely antisymmetized sum

$$
\begin{aligned}
&F^{\lambda\sigma}F_{\alpha\beta}\eta^{\mu\mu'}({\delta_\lambda}^\nu{\delta_\sigma}^\beta{\delta_{\mu'}}^\alpha - {\delta_\lambda}^\nu{\delta_\sigma}^\alpha{\delta_{\mu'}}^\beta - {\delta_\lambda}^\beta{\delta_\sigma}^\nu{\delta_{\mu'}}^\alpha + {\delta_\lambda}^\alpha{\delta_\sigma}^\nu{\delta_{\mu'}}^\beta + {\delta_\lambda}^\beta{\delta_\sigma}^\alpha{\delta_{\mu'}}^\nu - {\delta_\lambda}^\alpha{\delta_\sigma}^\beta{\delta_{\mu'}}^\nu) \\
&= F^{\nu\beta}F_{\mu'\beta}\eta^{\mu\mu'} - F^{\nu\alpha}F_{\alpha\mu'}\eta^{\mu\mu'} - F^{\beta\nu}F_{\mu'\beta}\eta^{\mu\mu'} + F^{\alpha\nu}F_{\alpha\mu'}\eta^{\mu\mu'} + F^{\beta\alpha}F_{\alpha\beta}\eta^{\mu\mu'}{\delta_{\mu'}}^\nu - F^{\alpha\beta}F_{\alpha\beta}\eta^{\mu\mu'}{\delta_{\mu'}}^\nu \\
&= 4F^{\nu\alpha}F_{\mu'\alpha}\eta^{\mu\mu'} + 2F^{\beta\alpha}F_{\alpha\beta}\eta^{\mu\mu'}{\delta_{\mu'}}^\nu \\
&= 4F^{\nu\alpha}{F^\mu}_\alpha + 2F^{\beta\alpha}F_{\alpha\beta}\eta^{\mu\nu}
\end{aligned}
$$

Combining these we have

$$
\begin{aligned}
T^{\mu\nu} &= -\frac{\epsilon_0}{8}\left(4F^{\alpha\mu}{F_\alpha}^\nu + 4F^{\nu\alpha}{F^\mu}_\alpha + 2F^{\beta\alpha}F_{\alpha\beta}\eta^{\mu\nu}\right) \\
&= \frac{\epsilon_0}{8}\left(-4F^{\alpha\mu}{F_\alpha}^\nu + 4F^{\alpha\mu}{F^\nu}_\alpha + 2F^{\alpha\beta}F_{\alpha\beta}\eta^{\mu\nu}\right)
\end{aligned}
$$

If by some miracle all the index manipulation worked out, we have

$$
T^{\mu\nu} = \epsilon_0\left(F^{\alpha\mu}{F^\nu}_\alpha + \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}\eta^{\mu\nu}\right) \tag{6}
$$

1.2.1 Justifying some of the steps.

For justification of some of the index manipulations of the F tensor components it is helpful to think back to the definitions in terms of four vector potentials

$$
\begin{aligned}
F &= \nabla \wedge A \\
&= \partial^\mu A^\nu \gamma_\mu \wedge \gamma_\nu \\
&= \partial_\mu A_\nu \gamma^\mu \wedge \gamma^\nu \\
&= \partial_\mu A^\nu \gamma^\mu \wedge \gamma_\nu \\
&= \partial^\mu A_\nu \gamma_\mu \wedge \gamma^\nu \\
&= \frac{1}{2}(\partial^\mu A^\nu - \partial^\nu A^\mu)\gamma_\mu \wedge \gamma_\nu \\
&= \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\gamma^\mu \wedge \gamma^\nu \\
&= \frac{1}{2}(\partial_\mu A^\nu - \partial^\nu A_\mu)\gamma^\mu \wedge \gamma_\nu \\
&= \frac{1}{2}(\partial^\mu A_\nu - \partial_\nu A^\mu)\gamma_\mu \wedge \gamma^\nu
\end{aligned}
$$

So with the shorthand

$$
\begin{aligned}
F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu \\
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\
{F_\mu}^\nu &= \partial_\mu A^\nu - \partial^\nu A_\mu \\
{F^\mu}_\nu &= \partial^\mu A_\nu - \partial_\nu A^\mu
\end{aligned}
$$

We have

$$
\begin{aligned}
F &= \frac{1}{2} F^{\mu\nu} \gamma_\mu \wedge \gamma_\nu \\
&= \frac{1}{2} F_{\mu\nu} \gamma^\mu \wedge \gamma^\nu \\
&= \frac{1}{2} {F_\mu}^\nu \gamma^\mu \wedge \gamma_\nu \\
&= \frac{1}{2} {F^\mu}_\nu \gamma_\mu \wedge \gamma^\nu
\end{aligned}
$$

In particular, and perhaps not obvious without the definitions handy, the following was used above

$$
{F^\mu}_\nu = -{F_\nu}^\mu
$$

1.3 The divergence.

What's our divergence in tensor form? This would be

$$
\nabla \cdot T(\gamma^\mu) = (\gamma^\alpha \partial_\alpha) \cdot (T^{\mu\nu} \gamma_\nu)
$$

So we have

$$
\nabla \cdot T(\gamma^\mu) = \partial_\nu T^{\mu\nu} \tag{7}
$$

Ignoring the ϵ_0 factor for now, chain rule gives us

$$
\begin{aligned}
&(\partial_\nu F^{\alpha\mu}) {F^\nu}_\alpha + F^{\alpha\mu} (\partial_\nu {F^\nu}_\alpha) + \frac{1}{2} (\partial_\nu F^{\alpha\beta}) F_{\alpha\beta} \eta^{\mu\nu} \\
&\quad = (\partial_\nu F^{\alpha\mu}) {F^\nu}_\alpha + {F_\alpha}^\mu (\partial_\nu F^{\nu\alpha}) + \frac{1}{2} (\partial_\nu F^{\alpha\beta}) F_{\alpha\beta} \eta^{\mu\nu}
\end{aligned}
$$

Only this center term is recognizable in terms of current since we have

$$
\nabla \cdot F = J/\epsilon_0 c
$$

Where the LHS is

$$
\begin{aligned}
\nabla \cdot F &= \gamma^\alpha \partial_\alpha \cdot \left(\frac{1}{2} F^{\mu\nu} \gamma_\mu \wedge \gamma_\nu \right) \\
&= \frac{1}{2} \partial_\alpha F^{\mu\nu} ({\delta^\alpha}_\mu \gamma_\nu - {\delta^\alpha}_\nu \gamma_\mu) \\
&= \partial_\mu F^{\mu\nu} \gamma_\nu
\end{aligned}
$$

So we have

$$
\begin{aligned}
\partial_\mu F^{\mu\nu} &= (J \cdot \gamma^\nu)/\epsilon_0 c \\
&= ((J^\alpha \gamma_\alpha) \cdot \gamma^\nu)/\epsilon_0 c \\
&= J^\nu/\epsilon_0 c
\end{aligned}
$$

Or

$$
\partial_\mu F^{\mu\nu} = J^\nu/\epsilon_0 c \tag{8}
$$

So we have

$$
\nabla \cdot T(\gamma^\mu) = \epsilon_0 \left((\partial_\nu F^{\alpha\mu}) {F^\nu}_\alpha + \frac{1}{2} (\partial_\nu F^{\alpha\beta}) F_{\alpha\beta} \eta^{\mu\nu} \right) + {F_\alpha}^\mu J^\alpha / c
$$

So, to get the expected result the remaining two derivative terms must somehow cancel. How to reduce these? Let's look at twice that

$$
\begin{aligned}
&2(\partial_\nu F^{\alpha\mu}) {F^\nu}_\alpha + (\partial_\nu F^{\alpha\beta}) F_{\alpha\beta} \eta^{\mu\nu} \\
&\quad = 2(\partial^\nu F^{\alpha\mu}) F_{\nu\alpha} + (\partial^\mu F^{\alpha\beta}) F_{\alpha\beta} \\
&\quad = (\partial^\nu F^{\alpha\mu})(F_{\nu\alpha} - F_{\alpha\nu}) + (\partial^\mu F^{\alpha\beta}) F_{\alpha\beta} \\
&\quad = (\partial^\alpha F^{\beta\mu}) F_{\alpha\beta} + (\partial^\beta F^{\mu\alpha}) F_{\alpha\beta} + (\partial^\mu F^{\alpha\beta}) F_{\alpha\beta} \\
&\quad = (\partial^\alpha F^{\beta\mu} + \partial^\beta F^{\mu\alpha} + \partial^\mu F^{\alpha\beta}) F_{\alpha\beta}
\end{aligned}
$$

Ah, there's the trivector term of Maxwell's equation hiding in there.

$$
\begin{aligned}
0 &= \nabla \wedge F \\
&= \gamma_\mu \partial^\mu \wedge \left(\frac{1}{2} F^{\alpha\beta} (\gamma_\alpha \wedge \gamma_\beta) \right) \\
&= \frac{1}{2} (\partial^\mu F^{\alpha\beta}) (\gamma_\mu \wedge \gamma_\alpha \wedge \gamma_\beta) \\
&= \frac{1}{3!} \left(\partial^\mu F^{\alpha\beta} + \partial^\alpha F^{\beta\mu} + \partial^\beta F^{\mu\alpha} \right) (\gamma_\mu \wedge \gamma_\alpha \wedge \gamma_\beta)
\end{aligned}
$$

Since this is zero, each component of this trivector must separately equal zero, and we have

$$\partial^\mu F^{\alpha\beta} + \partial^\alpha F^{\beta\mu} + \partial^\beta F^{\mu\alpha} = 0 \tag{9}$$

So, where $T^{\mu\nu}$ is defined by 6, the final result is

$$\partial_\nu T^{\mu\nu} = F^{\alpha\mu} J_\alpha / c \tag{10}$$

References

[Doran and Lasenby(2003)] C. Doran and A.N. Lasenby. *Geometric algebra for physicists*. Cambridge University Press New York, 2003.

[Joot(a)] Peeter Joot. Energy momentum tensor. "`http://sites.google.com/site/peeterjoot/math/energy_momentum_tensor.pdf`", a.

[Joot(b)] Peeter Joot. Lorentz force relation to the energy momentum tensor. "`http://sites.google.com/site/peeterjoot/math2009/stress_energy_lorentz.pdf`", b.