Relativistic Doppler formula.

Originally appeared at:
`http://sites.google.com/site/peeterjoot/math2009/frequencyTx.pdf`

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June 27, 2009 $RCSfile : frequencyTx.tex, v$ *Last Revision* : 1.4 *Date* : 2009/07/0704 : 29 : 36

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1. Transform of angular velocity four vector.

It was possible to derive the Lorentz boost matrix by requiring that the wave equation operator

$$\nabla^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2 \tag{1}$$

retain its form under linear transformation ([1]). Applying spatial Fourier transforms ([2]), one finds that solutions to the wave equation

$$\nabla^2 \psi(t, \mathbf{x}) = 0 \tag{2}$$

Have the form

$$\psi(t, \mathbf{x}) = \int A(\mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} d^3k \tag{3}$$

Provided that $\omega = \pm c|\mathbf{k}|$. Wave equation solutions can therefore be thought of as continuously weighted superpositions of constrained fundamental solutions

$$\psi = e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \tag{4}$$

$$c^2\mathbf{k}^2 = \omega^2 \tag{5}$$

The constraint on frequency and wave number has the look of a Lorentz square

$$\omega^2 - c^2\mathbf{k}^2 = 0 \tag{6}$$

Which suggests that in additional to the spacetime vector

$$X = (ct, \mathbf{x}) = x^\mu \gamma_\mu \tag{7}$$

evident in the wave equation fundamental solution, we also have a frequency-wavenumber four vector

$$K = (\omega/c, \mathbf{k}) = k^\mu \gamma_\mu \tag{8}$$

The pair of four vectors above allow the fundamental solutions to be put explicitly into covariant form

$$K \cdot X = \omega t - \mathbf{k} \cdot \mathbf{x} = k_\mu x^\mu \tag{9}$$

$$\psi = e^{-iK \cdot X} \tag{10}$$

Let's also examine the transformation properties of this fundamental solution, and see as a side effect that K has transforms appropriately as a four vector.

$$\begin{aligned}
0 &= \nabla^2 \psi(t, \mathbf{x}) \\
&= \nabla'^2 \psi(t', \mathbf{x}') \\
&= \nabla'^2 e^{i(\mathbf{x}' \cdot \mathbf{k}' - \omega' t')} \\
&= -\left(\frac{\omega'^2}{c^2} - \mathbf{k}'^2 \right) e^{i(\mathbf{x}' \cdot \mathbf{k}' - \omega' t')}
\end{aligned}$$

We therefore have the same form of frequency wave number constraint in the transformed frame (if we require that the wave function for light is unchanged under transformation)

$$\omega'^2 = c^2 \mathbf{k}'^2 \tag{11}$$

Writing this as

$$0 = \omega^2 - c^2 \mathbf{k}^2 = \omega'^2 - c^2 \mathbf{k}'^2 \tag{12}$$

singles out the Lorentz invariant nature of the $(\omega, \mathbf{k})$ pairing, and we conclude that this pairing does indeed transform as a four vector.

2. **Application of one dimensional boost.**

Having attempted to justify the four vector nature of the wave number vector K, now move on to application of a boost along the x-axis to this vector.

$$\begin{aligned}\begin{bmatrix}\omega' \\ ck'\end{bmatrix} &= \gamma \begin{bmatrix}1 & -\beta \\ -\beta & 1\end{bmatrix}\begin{bmatrix}\omega \\ ck\end{bmatrix} \\ &= \begin{bmatrix}\omega - vk \\ ck - \beta\omega\end{bmatrix}\end{aligned}$$

We can take ratios of the frequencies if we make use of the dependency between ω and k. Namely, $\omega = \pm ck$. We then have

$$\begin{aligned}\frac{\omega'}{\omega} &= \gamma(1 \mp \beta) \\ &= \frac{1 \mp \beta}{\sqrt{1-\beta^2}} \\ &= \frac{1 \mp \beta}{\sqrt{1-\beta}\sqrt{1+\beta}}\end{aligned}$$

For the positive angular frequency this is

$$\frac{\omega'}{\omega} = \frac{\sqrt{1-\beta}}{\sqrt{1+\beta}}$$

and for the negative frequency the reciprocal.

Deriving this with a Lorentz boost is much simpler than the time dilation argument in wikipedia doppler article ([3]). EDIT: Later found exactly the above boost argument in the wiki k-vector article ([4]).

What's missing here is putting this in a physical context properly with source and reciever frequencies spelled out. That would make this more than just math.

References

[1] Peeter Joot. Wave equation based Lorentz transformation derivation [online]. Available from: `http://sites.google.com/site/peeterjoot/geometric-algebra/lorentz.pdf`.

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[3] Wikipedia. Relativistic doppler effect — wikipedia, the free encyclopedia [online]. 2009. [Online; accessed 26-June-2009]. Available from: `http://en.wikipedia.org/w/index.php?title=Relativistic_Doppler_effect&oldid=298724264`.

[4] Wikipedia. Wave vector — wikipedia, the free encyclopedia [online]. 2009. [Online; accessed 30-June-2009]. Available from: http://en.wikipedia.org/w/index.php?title=Wave_vector&oldid=299450041.