Lorentz force from Lagrangian (non-covariant)

Originally appeared at:
`http://sites.google.com/site/peeterjoot/math2009/jackson12Dash9.pdf`

Peeter Joot — peeter.joot@gmail.com

Sept 22, 2009 $RCSfile : jackson12Dash9.tex, v$ $Last Revision : 1.2$ $Date : 2009/09/2303 : 43 : 07$

---

1. Motivation

Jackson [1] gives the Lorentz force non-covariant Lagrangian

$$L = -mc^2 \sqrt{1 - \mathbf{u}^2/c^2} + \frac{e}{c}\mathbf{u} \cdot \mathbf{A} - e\phi \tag{1}$$

and leaves it as an exercise for the reader to verify that this produces the Lorentz force law. Felt like trying this anew since I recall having trouble the first time I tried it (the covariant derivation was easier).

2. Guts

Jackson gives a tip to use the convective derivative (yet another name for the chain rule), and using this in the Euler Lagrange equations we have

$$\boldsymbol{\nabla}\mathcal{L} = \frac{d}{dt}\boldsymbol{\nabla}_{\mathbf{u}}\mathcal{L} = \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla}\right) \sigma_a \frac{\partial \mathcal{L}}{\partial \dot{x}^a} \tag{2}$$

where $\{\sigma_a\}$ is the spatial basis. The first order of business is calculating the gradient and conjugate momenta. For the latter we have

$$\begin{aligned}
\sigma_a \frac{\partial \mathcal{L}}{\partial \dot{x}^a} &= \sigma_a \left(-mc^2 \gamma \frac{1}{2}(-2)\dot{x}^a/c^2 + \frac{e}{c}A^a\right) \\
&= m\gamma \mathbf{u} + \frac{e}{c}\mathbf{A} \\
&\equiv \mathbf{p} + \frac{e}{c}\mathbf{A}
\end{aligned}$$

Applying the convective derivative we have

$$\frac{d}{dt}\sigma_a \frac{\partial \mathcal{L}}{\partial \dot{x}^a} = \frac{d\mathbf{p}}{dt} + \frac{e}{c}\frac{\partial \mathbf{A}}{\partial t} + \frac{e}{c}\mathbf{u} \cdot \boldsymbol{\nabla}\mathbf{A}$$

For the gradient we have

$$\sigma_a \frac{\partial \mathcal{L}}{x^a} = e\left(\frac{1}{c}\dot{x}^b \boldsymbol{\nabla} A^b - \boldsymbol{\nabla}\phi\right)$$

Rearranging 2 for this Lagrangian we have

$$\frac{d\mathbf{p}}{dt} = e\left(-\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \frac{1}{c}\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{A} + \frac{1}{c}\dot{x}^b\boldsymbol{\nabla}A^b\right)$$

The first two terms are the electric field

$$\mathbf{E} \equiv -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

So it remains to be shown that the remaining two equal $(\mathbf{u}/c)\times\mathbf{B} = (\mathbf{u}/c)\times(\boldsymbol{\nabla}\times\mathbf{A})$. Using the Hestenes notation using primes to denote what the gradient is operating on, we have

$$\begin{aligned}
\dot{x}^b\boldsymbol{\nabla}A^b - \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{A} &= \boldsymbol{\nabla}'\mathbf{u}\cdot\mathbf{A}' - \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{A} \\
&= -\mathbf{u}\cdot(\boldsymbol{\nabla}\wedge\mathbf{A}) \\
&= \frac{1}{2}\left((\boldsymbol{\nabla}\wedge\mathbf{A})\mathbf{u} - \mathbf{u}(\boldsymbol{\nabla}\wedge\mathbf{A})\right) \\
&= \frac{I}{2}\left((\boldsymbol{\nabla}\times\mathbf{A})\mathbf{u} - \mathbf{u}(\boldsymbol{\nabla}\times\mathbf{A})\right) \\
&= -I(\mathbf{u}\wedge\mathbf{B}) \\
&= -II(\mathbf{u}\times\mathbf{B}) \\
&= \mathbf{u}\times\mathbf{B}
\end{aligned}$$

I've used the Geometric Algebra identities I'm familiar with to regroup things, but this last bit can likely be done with index manipulation too. The exercise is complete, and we have from the Lagrangian

$$\frac{d\mathbf{p}}{dt} = e\left(\mathbf{E} + \frac{1}{c}\mathbf{u}\times\mathbf{B}\right) \tag{3}$$

References

[1] JD Jackson. *Classical Electrodynamics Wiley*. 2nd edition, 1975.