Levi-Civitica summation identity

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1 Motivation.

In [Byron and Fuller(1992)] it is left to the reader to show

$$\sum_k \epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}$$

2 A mechanical proof.

Although it's not mathematical, this is easy to prove, at least for 3D. The following perl code does the trick

```
#!/usr/bin/perl

$e{111} = 0  ; $e{112} = 0  ; $e{113} = 0 ;
$e{121} = 0  ; $e{122} = 0  ; $e{123} = 1 ;
$e{131} = 0  ; $e{132} = -1 ; $e{133} = 0 ;

$e{211} = 0  ; $e{212} = 0  ; $e{213} = -1 ;
$e{221} = 0  ; $e{222} = 0  ; $e{223} = 0 ;
$e{231} = 1  ; $e{232} = 0  ; $e{233} = 0 ;

$e{311} = 0  ; $e{312} = 1  ; $e{313} = 0 ;
$e{321} = -1 ; $e{322} = 0  ; $e{323} = 0 ;
$e{331} = 0  ; $e{332} = 0  ; $e{333} = 0 ;
```

```
$d{11} = 1 ; $d{12} = 0 ; $d{13} = 0 ;
$d{21} = 0 ; $d{22} = 1 ; $d{23} = 0 ;
$d{31} = 0 ; $d{32} = 0 ; $d{33} = 1 ;

# prove: \sum_k e_{ijk} e_{klm}
#   = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}
#print "$e{123} $e{113} $e{321}\n" ;

for ( my $i = 1 ; $i <= 3 ; $i++ ) {
 for ( my $j = 1 ; $j <= 3 ; $j++ ) {
  for ( my $l = 1 ; $l <= 3 ; $l++ ) {
   for ( my $m = 1 ; $m <= 3 ; $m++ ) {
    my $lhs = 0 ;
    my $rhs = $d{"${i}${l}"} * $d{"${j}${m}"}
            - $d{"${j}${l}"} * $d{"${i}${m}"} ;

    for ( my $k = 1 ; $k <= 3 ; $k++ ) {
     $lhs += $e{"${i}${j}${k}"} * $e{"${k}${l}${m}"} ;
    }

    if ( $rhs != $lhs ) {
     print 'ERROR: \\sum_{k=1}^{3} \\epsilon_{' .
           "${i}${j}k" .
           '} \\epsilon_{' .
           "k${l}${m}} = $lhs\n" ;

     print 'ERROR: \\delta_{' .
           "${i}${l}}" .  '\\delta_{' .
           "${j}${m}" .  '} - \\delta_{' .
           "${j}${l}" . '}\\delta_{' .
           "${i}${m}} = $rhs\n\n" ;
     } else {
      print "$lhs &= " .
            '\\sum_{k=1}^{3} \\epsilon_{' .
            "${i}${j}k" . '} \\epsilon_{' .
            "k${l}${m}} = " . '\\delta_{' .
            "${i}${l}}" . '\\delta_{' . "${j}${m}" .
            '} - \\delta_{' . "${j}${l}" .
            '}\\delta_{' . "${i}${m}}" .
            ' \\\\' . " \n" ;
} } } } }
```

The output produced has all the variations of indexes, such as

$$
\begin{aligned}
0 &= \sum_{k=1}^{3} \epsilon_{11k}\epsilon_{k11} = \delta_{11}\delta_{11} - \delta_{11}\delta_{11} \\
0 &= \sum_{k=1}^{3} \epsilon_{11k}\epsilon_{k12} = \delta_{11}\delta_{12} - \delta_{11}\delta_{12} \\
&\vdots \\
0 &= \sum_{k=1}^{3} \epsilon_{11k}\epsilon_{k33} = \delta_{13}\delta_{13} - \delta_{13}\delta_{13} \\
0 &= \sum_{k=1}^{3} \epsilon_{12k}\epsilon_{k11} = \delta_{11}\delta_{21} - \delta_{21}\delta_{11} \\
1 &= \sum_{k=1}^{3} \epsilon_{12k}\epsilon_{k12} = \delta_{11}\delta_{22} - \delta_{21}\delta_{12} \\
0 &= \sum_{k=1}^{3} \epsilon_{12k}\epsilon_{k13} = \delta_{11}\delta_{23} - \delta_{21}\delta_{13} \\
-1 &= \sum_{k=1}^{3} \epsilon_{12k}\epsilon_{k21} = \delta_{12}\delta_{21} - \delta_{22}\delta_{11} \\
&\vdots
\end{aligned}
$$

3 Proof using bivector dot product.

This identity can also be derived from an expansion of the bivector dot product in two different ways.

$$
\begin{aligned}
(\mathbf{e}_i \wedge \mathbf{e}_j) \cdot (\mathbf{e}_m \wedge \mathbf{e}_n) &= ((\mathbf{e}_i \wedge \mathbf{e}_j) \cdot \mathbf{e}_m) \cdot \mathbf{e}_n \\
&= (\mathbf{e}_i(\mathbf{e}_j \cdot \mathbf{e}_m) - \mathbf{e}_j(\mathbf{e}_i \cdot \mathbf{e}_m)) \cdot \mathbf{e}_n \\
&= (\mathbf{e}_i\delta_{jm} - \mathbf{e}_j\delta_{im}) \cdot \mathbf{e}_n \\
&= \delta_{in}\delta_{jm} - \delta_{jn}\delta_{im}
\end{aligned}
$$

Expressing the wedge product in terms duality, using the pseudoscalar $I = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$, we have

$$
(\mathbf{e}_i \wedge \mathbf{e}_j)\mathbf{e}_k = I\epsilon_{ijk}
$$

Or

$$\mathbf{e}_i \wedge \mathbf{e}_j = I \sum_k \epsilon_{ijk} \mathbf{e}_k$$

Then the bivector dot product is

$$\begin{aligned}
(\mathbf{e}_i \wedge \mathbf{e}_j) \cdot (\mathbf{e}_m \wedge \mathbf{e}_n) &= \left\langle I \sum_k \epsilon_{ijk} \mathbf{e}_k I \sum_p \epsilon_{mnp} \mathbf{e}_p \right\rangle \\
&= I^2 \sum_{k,p} \epsilon_{ijk} \epsilon_{mnp} \left\langle \mathbf{e}_k \mathbf{e}_p \right\rangle \\
&= - \sum_{k,p} \epsilon_{ijk} \epsilon_{mnp} \delta_{kp} \\
&= - \sum_k \epsilon_{ijk} \epsilon_{mnk}
\end{aligned}$$

Comparing the two expansions we have

$$\sum_k \epsilon_{ijk} \epsilon_{mnk} = \delta_{jn} \delta_{im} - \delta_{in} \delta_{jm}$$

Which is equivalent to the original identity (after an index switcheroo). Note both the dimension and metric dependencies in this proof.

References

[Byron and Fuller(1992)] F.W. Byron and R.W. Fuller. *Mathematics of Classical and Quantum Physics*. Dover Publications, 1992.