Lorentz boost of Lorentz force equations.

Peeter Joot   peeter.joot@gmail.com

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Contents

1   Motivation.

Reading of [Bohm(1996)] is a treatment of the Lorentz transform properties of the Lorentz force equation. This isn't clear to me without working through it myself, so do this.

I also have the urge to try this with the GA formulation of the Lorentz transformation. That may not end up being simpler if one works with the non-covariant form of the Lorentz force equation, but only trying it will tell.

2   Compare forms of the Lorentz Boost.

Working from the Geometric Algebra form of the Lorentz boost, show equivalence to the standard coordinate matrix form and the vector form from Bohm.

2.1 Exponential form.

Write the Lorentz boost of a four vector $x = x^\mu \gamma_\mu = ct\gamma_0 + x^k\gamma_k$ as

$$L(x) = e^{-\alpha\hat{v}/2}xe^{\alpha\hat{v}/2} \tag{1}$$

2.2 Invariance property.

A Lorentz transformation (boost or rotation) can be defined as those transformation that leave the four vector square unchanged.

Following [Doran and Lasenby(2003)], work with a $+---$ metric signature ($1 = \gamma_0^2 = -\gamma_k^2$), and $\sigma_k = \gamma_k\gamma_0$. Our four vector square in this representation has the familiar invariant form

$$
\begin{aligned}
x^2 &= (ct\gamma_0 + x^m\gamma_m)(ct\gamma_0 + x^k\gamma_k) \\
&= (ct\gamma_0 + x^m\gamma_m)\gamma_0^2(ct\gamma_0 + x^k\gamma_k) \\
&= (ct + x^m\sigma_m)(ct - x^k\sigma_k) \\
&= (ct + \mathbf{x})(ct - \mathbf{x}) \\
&= (ct)^2 - \mathbf{x}^2
\end{aligned}
$$

and we expect this of the Lorentz boost of equation 1. To verify we have

$$
\begin{aligned}
L(x)^2 &= e^{-\alpha\hat{v}/2}xe^{\alpha\hat{v}/2}e^{-\alpha\hat{v}/2}xe^{\alpha\hat{v}/2} \\
&= e^{-\alpha\hat{v}/2}xxe^{\alpha\hat{v}/2} \\
&= x^2e^{-\alpha\hat{v}/2}e^{\alpha\hat{v}/2} \\
&= x^2
\end{aligned}
$$

2.3 Sign of the rapidity angle.

The factor α will be the rapidity angle, but what sign do we want for a boost along the positive $\hat{v}$ direction?

Dropping to coordinates is an easy way to determine the sign convention in effect. Write $\hat{v} = \sigma_1$

$$
\begin{aligned}
L(x) &= e^{-\alpha\hat{v}/2}xe^{\alpha\hat{v}/2} \\
&= (\cosh(\alpha/2) - \sigma_1\sinh(\alpha/2))(x^0\gamma_0 + x^1\gamma_1 + x^2\gamma_2 + x^3\gamma_3)(\cosh(\alpha/2) + \sigma_1\sinh(\alpha/2))
\end{aligned}
$$

σ_1 commutes with γ_2 and γ_3 and anticommutes otherwise, so we have

$$L(x) = (x^2 \gamma_2 + x^3 \gamma_3)e^{-\alpha \hat{\mathbf{v}}/2} e^{\alpha \hat{\mathbf{v}}/2} + (x^0 \gamma_0 + x^1 \gamma_1)e^{\alpha \hat{\mathbf{v}}}$$
$$= x^2 \gamma_2 + x^3 \gamma_3 + (x^0 \gamma_0 + x^1 \gamma_1)e^{\alpha \hat{\mathbf{v}}}$$
$$= x^2 \gamma_2 + x^3 \gamma_3 + (x^0 \gamma_0 + x^1 \gamma_1)(\cosh(\alpha) + \sigma_1 \sinh(\alpha))$$

Expanding out just the $0, 1$ terms changed by the transformation we have

$$(x^0 \gamma_0 + x^1 \gamma_1)(\cosh(\alpha) + \sigma_1 \sinh(\alpha))$$
$$= x^0 \gamma_0 \cosh(\alpha) + x^1 \gamma_1 \cosh(\alpha) + x^0 \gamma_0 \sigma_1 \sinh(\alpha) + x^1 \gamma_1 \sigma_1 \sinh(\alpha)$$
$$= x^0 \gamma_0 \cosh(\alpha) + x^1 \gamma_1 \cosh(\alpha) + x^0 \gamma_0 \gamma_1 \gamma_0 \sinh(\alpha) + x^1 \gamma_1 \gamma_1 \gamma_0 \sinh(\alpha)$$
$$= x^0 \gamma_0 \cosh(\alpha) + x^1 \gamma_1 \cosh(\alpha) - x^0 \gamma_1 \sinh(\alpha) - x^1 \gamma_0 \sinh(\alpha)$$
$$= \gamma_0(x^0 \cosh(\alpha) - x^1 \sinh(\alpha)) + \gamma_1(x^1 \cosh(\alpha) - x^0 \sinh(\alpha))$$

Writing $x^{\mu'} = L(x) \cdot \gamma^\mu$, and $x^\mu = x \cdot \gamma^\mu$, and a substitution of $\cosh(\alpha) = 1/\sqrt{1 - \mathbf{v}^2/c^2}$, and $\alpha \hat{\mathbf{v}} = \tanh^{-1}(\mathbf{v}/c)$, we have the traditional coordinate expression for the one directional Lorentz boost

$$\begin{bmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{bmatrix} = \begin{bmatrix} \cosh\alpha & -\sinh\alpha & 0 & 0 \\ -\sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \qquad (2)$$

Performing this expansion showed initially showed that I had the wrong sign for α in the exponentials and I went back and adjusted it all accordingly.

2.4 Expanding out the Lorentz boost for projective and rejective directions.

Two forms of Lorentz boost representations have been compared above. An additional one is used in the Bohm text (a vector form of the Lorentz transformation not using coordinates). Let's see if we can derive that from the exponential form.

Start with computation of components of a four vector relative to an observer timelike unit vector γ_0.

$$x = x \gamma_0 \gamma_0$$
$$= (x \gamma_0)\gamma_0$$
$$= (x \cdot \gamma_0 + x \wedge \gamma_0)\gamma_0$$

For the spatial vector factor above write $\mathbf{x} = x \wedge \gamma_0$, for

$$
\begin{aligned}
x &= (x \cdot \gamma_0)\gamma_0 + \mathbf{x}\gamma_0 \\
&= (x \cdot \gamma_0)\gamma_0 + \mathbf{x}\hat{\mathbf{v}}\hat{\mathbf{v}}\gamma_0 \\
&= (x \cdot \gamma_0)\gamma_0 + (\mathbf{x} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}}\gamma_0 + (\mathbf{x} \wedge \hat{\mathbf{v}})\hat{\mathbf{v}}\gamma_0
\end{aligned}
$$

We have the following commutation relations for the various components

$$
\begin{aligned}
\hat{\mathbf{v}}(\gamma_0) &= -\gamma_0\hat{\mathbf{v}} \\
\hat{\mathbf{v}}(\hat{\mathbf{v}}\gamma_0) &= -(\hat{\mathbf{v}}\gamma_0)\hat{\mathbf{v}} \\
\hat{\mathbf{v}}((\mathbf{x} \wedge \hat{\mathbf{v}})\hat{\mathbf{v}}\gamma_0) &= ((\mathbf{x} \wedge \hat{\mathbf{v}})\hat{\mathbf{v}}\gamma_0)\hat{\mathbf{v}}
\end{aligned}
$$

For a four vector u that commutes with $\hat{\mathbf{v}}$ we have $e^{-\alpha\hat{\mathbf{v}}/2}u = ue^{-\alpha\hat{\mathbf{v}}/2}$, and if it anticommutes we have the conjugate relation $e^{-\alpha\hat{\mathbf{v}}/2}u = ue^{\alpha\hat{\mathbf{v}}/2}$. This gives us

$$
L(x) = (\mathbf{x} \wedge \hat{\mathbf{v}})\hat{\mathbf{v}}\gamma_0 + \left((x \cdot \gamma_0)\gamma_0 + (\mathbf{x} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}}\gamma_0\right)e^{\alpha\hat{\mathbf{v}}}
$$

Now write the exponential as a scalar and spatial vector sum

$$
\begin{aligned}
e^{\alpha\hat{\mathbf{v}}} &= \cosh\alpha + \hat{\mathbf{v}}\sinh\alpha \\
&= \gamma(1 + \hat{\mathbf{v}}\tanh\alpha) \\
&= \gamma(1 + \hat{\mathbf{v}}\beta) \\
&= \gamma(1 + \mathbf{v}/c)
\end{aligned}
$$

Expanding out the exponential product above, also writing $x^0 = ct = x \cdot \gamma_0$, we have

$$
\begin{aligned}
&(x^0\gamma_0 + (\mathbf{x} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}}\gamma_0)e^{\alpha\hat{\mathbf{v}}} \\
&= \gamma(x^0\gamma_0 + (\mathbf{x} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}}\gamma_0)(1 + \mathbf{v}/c) \\
&= \gamma(x^0\gamma_0 + (\mathbf{x} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}}\gamma_0 + x^0\gamma_0\mathbf{v}/c + (\mathbf{x} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}}\gamma_0\mathbf{v}/c)
\end{aligned}
$$

So for the total Lorentz boost in vector form we have

$$
L(x) = (\mathbf{x} \wedge \hat{\mathbf{v}})\hat{\mathbf{v}}\gamma_0 + \gamma\left(x^0 - \mathbf{x} \cdot \frac{\mathbf{v}}{c}\right)\gamma_0 + \gamma\left(\mathbf{x} \cdot \frac{1}{\mathbf{v}/c} - x^0\right)\frac{\mathbf{v}}{c}\gamma_0 \qquad (3)
$$

Now a visual inspection shows that this does match equation (15-12) from the text:

$$x' = x - (\hat{v} \cdot x)\hat{v} + \frac{(\hat{v} \cdot x)\hat{v} - vt}{\sqrt{1 - (v^2/c^2)}} \tag{4}$$

$$t' = \frac{t - (v \cdot x)/c^2}{\sqrt{1 - (v^2/c^2)}} \tag{5}$$

but the equivalence of these is perhaps not so obvious without familiarity with the GA constructs.

2.5 differential form.

Bohm utilizes a vector differential form of the Lorentz transformation for both the spacetime and energy-momentum vectors. From equation 4 we can derive the expressions used. In particular for the transformed spatial component we have

$$\begin{aligned}
x' &= x + \gamma\left(-(\hat{v} \cdot x)\hat{v}\frac{1}{\gamma} + (\hat{v} \cdot x)\hat{v} - vt\right) \\
&= x + \gamma\left((\hat{v} \cdot x)\hat{v}\left(1 - \frac{1}{\gamma}\right) - vt\right) \\
&= x + (\gamma - 1)(\hat{v} \cdot x)\hat{v} - \gamma vt
\end{aligned}$$

So in differential vector form we have

$$d x' = d x + (\gamma - 1)(\hat{v} \cdot d x)\hat{v} - \gamma v dt \tag{6}$$

$$dt' = \gamma(dt - (v \cdot d x)/c^2) \tag{7}$$

and by analogy with $dx^0 = cdt \rightarrow dE/c$, and $d x \rightarrow d p$, we also have the energy momentum transformation

$$d p' = d p + (\gamma - 1)(\hat{v} \cdot d p)\hat{v} - \gamma v dE/c^2 \tag{8}$$

$$dE' = \gamma(dE - v \cdot d p) \tag{9}$$

Reflecting on these forms of the Lorentz transformation, they are quite natural ways to express the vector results. The terms with γ factors are exactly what we are used to in the coordinate representation (transformation of only the time component and the projection of the spatial vector in the velocity direction), while the -1 part of the $(\gamma - 1)$ term just subtracts off the projection unaltered, leaving $d x - (d x \cdot \hat{v})\hat{v} = (d x \wedge \hat{v})\hat{v}$, the rejection from the $\hat{v}$ direction.

3   Lorentz force transformation.

Preliminaries out of the way, now we want to examine the tranform of the electric and magnetic field as used in the Lorentz force equation. In csg units as in the text we have

$$\frac{d\mathbf{p}}{dt} = q\left(\boldsymbol{\mathcal{E}} + \frac{\mathbf{v}}{c} \times \boldsymbol{\mathcal{H}}\right) \tag{10}$$

$$\frac{dE}{dt} = q\boldsymbol{\mathcal{E}} \cdot \mathbf{v} \tag{11}$$

After writing this in differential form

$$d\mathbf{p} = q\left(\boldsymbol{\mathcal{E}}dt + \frac{d\mathbf{x}}{c} \times \boldsymbol{\mathcal{H}}\right) \tag{12}$$

$$dE = q\boldsymbol{\mathcal{E}} \cdot d\mathbf{x} \tag{13}$$

and the transformed variation of this equation, also in differential form

$$d\mathbf{p}' = q\left(\boldsymbol{\mathcal{E}}'dt' + \frac{d\mathbf{x}'}{c} \times \boldsymbol{\mathcal{H}}'\right) \tag{14}$$

$$dE' = q\boldsymbol{\mathcal{E}}' \cdot d\mathbf{x}' \tag{15}$$

A brute force insertion of the transform results of equations 6, and 8 into these is performed. This is mostly a mess of algebra.

While the Bohm book covers some of this, other parts are left for the reader. Do the whole thing here as an exersize.

3.1   Transforming the Lorentz power equation.

Let's start with the energy rate equation in its entirety without interleaving the momentum calculation.

$$\frac{1}{q}dE' = \boldsymbol{\mathcal{E}}' \cdot d\mathbf{x}'$$
$$= \boldsymbol{\mathcal{E}}' \cdot \left(d\mathbf{x} + (\gamma - 1)(\hat{\mathbf{V}} \cdot d\mathbf{x})\hat{\mathbf{V}} - \gamma\mathbf{V}dt\right)$$
$$= \boldsymbol{\mathcal{E}}' \cdot d\mathbf{x} + (\gamma - 1)(\hat{\mathbf{V}} \cdot d\mathbf{x})\boldsymbol{\mathcal{E}}' \cdot \hat{\mathbf{V}} - \gamma\boldsymbol{\mathcal{E}}' \cdot \mathbf{V}dt$$
$$\frac{1}{q}\gamma(dE - \mathbf{V} \cdot d\mathbf{p}) =$$
$$\gamma\boldsymbol{\mathcal{E}} \cdot d\mathbf{x} - \gamma\mathbf{V} \cdot \left(\boldsymbol{\mathcal{E}}dt + \frac{d\mathbf{x}}{c} \times \boldsymbol{\mathcal{H}}\right) =$$
$$\gamma\boldsymbol{\mathcal{E}} \cdot d\mathbf{x} - \gamma\mathbf{V} \cdot \boldsymbol{\mathcal{E}}dt - \gamma\frac{1}{c}d\mathbf{x} \cdot (\boldsymbol{\mathcal{H}} \times \mathbf{V}) =$$

Grouping dt and $d\mathbf{x}$ terms we have

$$0 = d\mathbf{x} \cdot \left(\boldsymbol{\mathcal{E}}' + (\gamma - 1)\hat{\mathbf{V}}(\boldsymbol{\mathcal{E}}' \cdot \hat{\mathbf{V}}) - \gamma\boldsymbol{\mathcal{E}} + \gamma(\boldsymbol{\mathcal{H}} \times \mathbf{V}/c) \right) + dt\gamma\mathbf{V} \cdot (\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{E}}')$$

Now the argument is that both the dt and $d\mathbf{x}$ factors must separately equal zero. Assuming that for now (but come back to this and think it through), and writing $\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}_{\parallel} + \boldsymbol{\mathcal{E}}_{\perp}$ for the projective and rejective components of the field relative to the boost direction $\mathbf{V}$ (same for $\boldsymbol{\mathcal{H}}$ and the transformed fields) we have from the dt term

$$\begin{aligned}
0 &= \mathbf{V} \cdot (\boldsymbol{\mathcal{E}}_{\parallel} + \boldsymbol{\mathcal{E}}_{\perp} - \boldsymbol{\mathcal{E}}'_{\parallel} - \boldsymbol{\mathcal{E}}'_{\perp}) \\
&= \mathbf{V} \cdot (\boldsymbol{\mathcal{E}}_{\parallel} - \boldsymbol{\mathcal{E}}'_{\parallel})
\end{aligned}$$

So we can conclude

$$\boldsymbol{\mathcal{E}}'_{\parallel} = \boldsymbol{\mathcal{E}}_{\parallel}$$

Now from the $d\mathbf{x}$ coefficient, we have

$$0 = \boldsymbol{\mathcal{E}}'_{\parallel} + \boldsymbol{\mathcal{E}}'_{\perp} + (\gamma - 1)\hat{\mathbf{V}}(\boldsymbol{\mathcal{E}}'_{\parallel} \cdot \hat{\mathbf{V}}) - \gamma\boldsymbol{\mathcal{E}}_{\parallel} - \gamma\boldsymbol{\mathcal{E}}_{\perp} + \gamma(\boldsymbol{\mathcal{H}}_{\perp} \times \mathbf{V}/c)$$

$$= \left(\underbrace{\boldsymbol{\mathcal{E}}'_{\parallel} - \hat{\mathbf{V}}(\boldsymbol{\mathcal{E}}'_{\parallel} \cdot \hat{\mathbf{V}})}_{\boldsymbol{\mathcal{E}}'_{\parallel} - \boldsymbol{\mathcal{E}}'_{\parallel}} \right) + \boldsymbol{\mathcal{E}}'_{\perp} - \gamma \left(\underbrace{\boldsymbol{\mathcal{E}}_{\parallel} - \hat{\mathbf{V}}(\boldsymbol{\mathcal{E}}'_{\parallel} \cdot \hat{\mathbf{V}})}_{\boldsymbol{\mathcal{E}}_{\parallel} - \boldsymbol{\mathcal{E}}_{\parallel}} \right) - \gamma\boldsymbol{\mathcal{E}}_{\perp} + \gamma(\boldsymbol{\mathcal{H}}_{\perp} \times \mathbf{V}/c)$$

This now completely specifies the transformation properties of the electric field under a $\mathbf{V}$ boost

$$\boldsymbol{\mathcal{E}}'_{\perp} = \gamma \left(\boldsymbol{\mathcal{E}}_{\perp} + \frac{\mathbf{V}}{c} \times \boldsymbol{\mathcal{H}}_{\perp} \right) \tag{16}$$

$$\boldsymbol{\mathcal{E}}'_{\parallel} = \boldsymbol{\mathcal{E}}_{\parallel} \tag{17}$$

(it also confirms the typos in the text).

3.2 Transforming the Lorentz momentum equation.

Now we do the exersize for the reader part, and express the transformed momentum differential of equation 14 in terms of 6

$$\frac{1}{q}d\mathbf{p}' = \boldsymbol{\mathcal{E}}'dt' + \frac{d\mathbf{x}'}{c} \times \boldsymbol{\mathcal{H}}'$$
$$= \gamma\boldsymbol{\mathcal{E}}'dt - \gamma\boldsymbol{\mathcal{E}}'(\mathbf{V}\cdot d\mathbf{x})/c^2 + d\mathbf{x}\times\boldsymbol{\mathcal{H}}'/c + (\gamma-1)(\hat{\mathbf{V}}\cdot d\mathbf{x})\hat{\mathbf{V}}\times\boldsymbol{\mathcal{H}}'/c - \gamma\mathbf{V}\times\boldsymbol{\mathcal{H}}'/cdt$$

Now for the LHS using 8 and 12 we have

$$\frac{1}{q}d\mathbf{p}' = d\mathbf{p}/q + (\gamma-1)(\hat{\mathbf{V}}\cdot d\mathbf{p}/q)\hat{\mathbf{V}} - \gamma\mathbf{V}dE/qc^2$$

$$= \boldsymbol{\mathcal{E}}dt + \frac{d\mathbf{x}}{c}\times\boldsymbol{\mathcal{H}} + (\gamma-1)(\hat{\mathbf{V}}\cdot\boldsymbol{\mathcal{E}}dt + \hat{\mathbf{V}}\cdot(d\mathbf{x}\times\boldsymbol{\mathcal{H}}/c))\hat{\mathbf{V}} - \gamma\mathbf{V}(\boldsymbol{\mathcal{E}}\cdot d\mathbf{x})/c^2$$

$$= \boldsymbol{\mathcal{E}}dt + \frac{d\mathbf{x}}{c}\times\boldsymbol{\mathcal{H}} + (\gamma-1)(\hat{\mathbf{V}}\cdot\boldsymbol{\mathcal{E}})\hat{\mathbf{V}}dt + (\gamma-1)(d\mathbf{x}\cdot(\boldsymbol{\mathcal{H}}\times\hat{\mathbf{V}}/c))\hat{\mathbf{V}} - \gamma\mathbf{V}(\boldsymbol{\mathcal{E}}\cdot d\mathbf{x})/c^2$$

Combining these and grouping by dt and $d\mathbf{x}$ we have

$$dt\left(-(\boldsymbol{\mathcal{E}} - (\hat{\mathbf{V}}\cdot\boldsymbol{\mathcal{E}})\hat{\mathbf{V}}) + \gamma(\boldsymbol{\mathcal{E}}' - (\hat{\mathbf{V}}\cdot\boldsymbol{\mathcal{E}})\hat{\mathbf{V}}) - \gamma\mathbf{V}\times\boldsymbol{\mathcal{H}}'/c\right)$$
$$= \frac{\gamma}{c^2}\left(\boldsymbol{\mathcal{E}}'(\mathbf{V}\cdot d\mathbf{x}) - \mathbf{V}(\boldsymbol{\mathcal{E}}\cdot d\mathbf{x})\right) + \frac{d\mathbf{x}}{c}\times(\boldsymbol{\mathcal{H}}-\boldsymbol{\mathcal{H}}') + \frac{\gamma-1}{c}\left((d\mathbf{x}\cdot(\boldsymbol{\mathcal{H}}\times\hat{\mathbf{V}}))\hat{\mathbf{V}} - (\hat{\mathbf{V}}\cdot d\mathbf{x})(\hat{\mathbf{V}}\times\boldsymbol{\mathcal{H}}')\right)$$

What a mess, and this is after some initial grouping! From the power result we have $\hat{\mathbf{V}}\cdot\boldsymbol{\mathcal{E}} = \hat{\mathbf{V}}\cdot\boldsymbol{\mathcal{E}}'$ so we can write the LHS of this mess as

$$dt\left(-(\boldsymbol{\mathcal{E}} - (\hat{\mathbf{V}}\cdot\boldsymbol{\mathcal{E}})\hat{\mathbf{V}}) + \gamma(\boldsymbol{\mathcal{E}}' - (\hat{\mathbf{V}}\cdot\boldsymbol{\mathcal{E}})\hat{\mathbf{V}}) - \gamma\mathbf{V}\times\boldsymbol{\mathcal{H}}'/c\right)$$
$$= dt\left(-(\boldsymbol{\mathcal{E}} - (\hat{\mathbf{V}}\cdot\boldsymbol{\mathcal{E}})\hat{\mathbf{V}}) + \gamma(\boldsymbol{\mathcal{E}}' - (\hat{\mathbf{V}}\cdot\boldsymbol{\mathcal{E}}')\hat{\mathbf{V}}) - \gamma\mathbf{V}\times\boldsymbol{\mathcal{H}}'/c\right)$$
$$= dt\left(-\boldsymbol{\mathcal{E}}_\perp + \gamma\boldsymbol{\mathcal{E}}'_\perp - \gamma\mathbf{V}\times\boldsymbol{\mathcal{H}}'/c\right)$$
$$= dt\left(-\boldsymbol{\mathcal{E}}_\perp + \gamma\boldsymbol{\mathcal{E}}'_\perp - \gamma\mathbf{V}\times\boldsymbol{\mathcal{H}}'_\perp/c\right)$$

If this can separately equal zero independent of the $d\mathbf{x}$ terms we have

$$\boldsymbol{\mathcal{E}}_\perp = \gamma\left(\boldsymbol{\mathcal{E}}'_\perp - \frac{\mathbf{V}}{c}\times\boldsymbol{\mathcal{H}}'_\perp\right) \tag{18}$$

Constrast this to the result for $\boldsymbol{\mathcal{E}}'_\perp$ in the first of 16. It differs only by a sign which has an intuitive relativistic (anti)symmetry that isn't entirely unsupprising. If a boost along $\mathbf{V}$ takes $\boldsymbol{\mathcal{E}}$ to $\boldsymbol{\mathcal{E}}'$, then an boost with opposing direction makes sense for the reverse.

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Despite being reasonable seeming, a relation like $\mathcal{H}_\parallel = \mathcal{H}'_\parallel$ was expected ... does that follow from this somehow? Perhaps things will become more clear after examining the mess on the RHS involving all the $d\mathbf{x}$ terms?

The first part of this looks amenable to some algebraic manipulation. Using $(\mathcal{E}' \wedge \mathbf{V}) \cdot d\mathbf{x} = \mathcal{E}'(\mathbf{V} \cdot d\mathbf{x}) - \mathbf{V}(\mathcal{E}' \cdot d\mathbf{x})$, we have

$$\mathcal{E}'(\mathbf{V} \cdot d\mathbf{x}) - \mathbf{V}(\mathcal{E} \cdot d\mathbf{x}) = (\mathcal{E}' \wedge \mathbf{V}) \cdot d\mathbf{x} + \mathbf{V}(\mathcal{E}' \cdot d\mathbf{x}) - \mathbf{V}(\mathcal{E} \cdot d\mathbf{x})$$
$$= (\mathcal{E}' \wedge \mathbf{V}) \cdot d\mathbf{x} + \mathbf{V}((\mathcal{E}' - \mathcal{E}) \cdot d\mathbf{x})$$

and

$$(\mathcal{E}' \wedge \mathbf{V}) \cdot d\mathbf{x} = \left\langle (\mathcal{E}' \wedge \mathbf{V}) d\mathbf{x} \right\rangle_1$$
$$= \left\langle i(\mathcal{E}' \times \mathbf{V}) d\mathbf{x} \right\rangle_1$$
$$= \left\langle i((\mathcal{E}' \times \mathbf{V}) \wedge d\mathbf{x}) \right\rangle_1$$
$$= \left\langle i^2((\mathcal{E}' \times \mathbf{V}) \times d\mathbf{x}) \right\rangle_1$$
$$= d\mathbf{x} \times (\mathcal{E}' \times \mathbf{V})$$

Putting things back together, does it improve things?

$$0 = d\mathbf{x} \times \left(\gamma \left(\mathcal{E}' \times \frac{\mathbf{V}}{c} \right) + (\mathcal{H} - \mathcal{H}') \right)$$
$$+ \frac{\gamma}{c} \mathbf{V}((\mathcal{E}' - \mathcal{E}) \cdot d\mathbf{x})$$
$$+ (\gamma - 1) \left((d\mathbf{x} \cdot (\mathcal{H} \times \hat{\mathbf{V}})) \hat{\mathbf{V}} - (\hat{\mathbf{V}} \cdot d\mathbf{x})(\hat{\mathbf{V}} \times \mathcal{H}') \right)$$

Perhaps the last bit can be factored into $d\mathbf{x}$ crossed with some function of $\mathcal{H} - \mathcal{H}'$?

References

[Bohm(1996)] D. Bohm. *The special theory of relativity*. Routledge, 1996.

[Doran and Lasenby(2003)] C. Doran and A.N. Lasenby. *Geometric algebra for physicists*. Cambridge University Press New York, 2003.