Lagrangian and Euler-Lagrange equation evaluation for the spherical N-pendulum problem.

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Abstract. The dynamics of chain like objects can be idealized as a multiple pendulum, treating the system as a set of point masses, joined by rigid massless connecting rods, and frictionless pivots. The double planar pendulum and single mass spherical pendulum problems are well treated in Lagrangian physics texts, but due to complexity a similar treatment of the spherical N-pendulum problem is not pervasive. We show that this problem can be tackled in a direct fashion, even in the general case with multiple masses and no planar constraints. A matrix factorization of the kinetic energy into allows an explicit and compact specification of the Lagrangian. Once that is obtained the equations of motion for this generalized pendulum system follow directly.

1. Introduction.

Derivation of the equations of motion for a planar motion constrained double pendulum system and a single spherical pendulum system are given as problems or examples in many texts covering Lagrangian mechanics. Setup of the Lagrangian, particularly an explicit specification of the system kinetic energy, is the difficult aspect of the multiple mass pendulum problem. Each mass in the system introduces additional interaction coupling terms, complicating the kinetic energy specification. In this paper, we use matrix algebra to determine explicitly the Lagrangian for the spherical N pendulum system, and to evaluate the Euler-Lagrange equations for the system.

It is well known that the general specification of the kinetic energy for a system of independent point masses takes the form of a symmetric quadratic form [1] [2]. However, actually calculating that energy explicitly for the general N-pendulum is likely thought too pedantic for even the most punishing instructor to inflict on students as a problem or example.

Given a 3×1 coordinate vector of velocity components for each mass relative to the position of the mass it is connected to, we can factor this as a $(3 \times 2)(2 \times 1)$ product of matrices where the 2×1 matrix is a vector of angular velocity components in the spherical polar representation. The remaining matrix factor contains all the trigonometric dependence. Such a grouping can be used to tidily separate the kinetic energy into an explicit quadratic form, sandwiching a symmetric matrix between two vectors of generalized velocity coordinates.

This paper is primarily a brute force and direct attack on the problem. It contains no new science, only a systematic treatment of a problem that is omitted from mechanics texts, yet conceptually simple enough to deserve treatment.

The end result of this paper is a complete and explicit specification of the Lagrangian and evaluation of the Euler-Lagrange equations for the chain-like N spherical pendulum system. While this end result is essentially nothing more than a non-linear set of coupled differential equations, it is believed that the approach used to obtain it has some elegance. Grouping all the rotational terms of the kinetic into a symmetric kernel appears to be a tidy way to tackle multiple discrete mass problems. At the very least, the calculation performed can show that a problem perhaps thought to be too messy for a homework exersize yields nicely to an organized and systematic attack.

2. **Diving right in.**

We make the simplifying assumptions of point masses, rigid massless connecting rods, and frictionless pivots.

2.1. ***Single spherical pendulum.***

Using polar angle θ and azimuthal angle ϕ, and writing $S_\theta = \sin\theta$, $C_\phi = \cos\phi$ and so forth, we have for the coordinate vector on the unit sphere

$$\hat{\mathbf{r}} = \begin{bmatrix} C_\phi S_\theta \\ S_\phi S_\theta \\ C_\theta \end{bmatrix}. \tag{1}$$

The Lagrangian for the pendulum is then

$$\mathcal{L} = \frac{1}{2} ml \dot{\hat{\mathbf{r}}}^T \dot{\hat{\mathbf{r}}} - mglC_\theta. \tag{2}$$

This is somewhat unsatisfying since the unit vector derivatives have not been evaluated. Doing so we get

$$\dot{\hat{\mathbf{r}}} = \begin{bmatrix} C_\phi C_\theta \dot{\theta} - S_\phi S_\theta \dot{\phi} \\ S_\phi C_\theta \dot{\theta} + C_\phi S_\theta \dot{\phi} \\ -S_\theta \dot{\theta} \end{bmatrix}. \tag{3}$$

This however, is an ugly beastie to take the norm of as is. It is straightforward to show that this norm is just

$$\dot{\hat{\mathbf{r}}}^\mathrm{T} \dot{\hat{\mathbf{r}}} = \dot{\theta}^2 + S_\theta^2 \dot{\phi}^2, \tag{4}$$

however, the brute force multiplication that leads to this result is not easily generalized to the multiple pendulum problem. Instead of actually expanding this now, lets defer that until later and instead write for a coordinate vector of angular velocity components

$$\Omega = \begin{bmatrix} \dot{\theta} \\ \dot{\phi} \end{bmatrix}. \tag{5}$$

Now the unit polar derivative 3 can be factored as

$$\dot{\mathbf{r}} = A^{\mathrm{T}}\Omega \tag{6a}$$

$$A = \begin{bmatrix} C_\phi C_\theta & S_\phi C_\theta & -S_\theta \\ -S_\phi S_\theta & C_\phi S_\theta & 0 \end{bmatrix}. \tag{6b}$$

Our Lagrangian now takes the explicit form

$$\mathcal{L} = \frac{1}{2} ml\Omega^{\mathrm{T}} A A^{\mathrm{T}} \Omega - mglC_\theta \tag{7a}$$

$$AA^{\mathrm{T}} = \begin{bmatrix} 1 & 0 \\ 0 & S_\theta^2 \end{bmatrix}. \tag{7b}$$

2.2. ***N spherical pendulum.***

2.2.1. The Lagrangian.

The position vector for each particle can be expressed relative to the mass it is connected to (or the origin for the first particle), as in

$$z_k = z_{k-1} + l_k \hat{\mathbf{r}}_k \tag{8a}$$

$$\hat{\mathbf{r}}_k = {A_k}^{\mathrm{T}} \dot{\mathbf{\Theta}}_k \tag{8b}$$

$$A_k = \begin{bmatrix} C_{\phi_k} C_{\theta_k} & S_{\phi_k} C_{\theta_k} & -S_{\theta_k} \\ -S_{\phi_k} S_{\theta_k} & C_{\phi_k} S_{\theta_k} & 0 \end{bmatrix} \tag{8c}$$

$$\mathbf{\Theta}_k = \begin{bmatrix} \theta_k \\ \phi_k \end{bmatrix}. \tag{8d}$$

Now, the relative velocity differential can be written utilizing these factors

$$(\dot{z}_k - \dot{z}_{k-1})^2 = l_k^2 \dot{\mathbf{\Theta}}_k^{\mathrm{T}} A_k A_k^{\mathrm{T}} \dot{\mathbf{\Theta}}_k. \tag{9}$$

Observe that the inner product is symmetric since $(A_k A_k^{\mathrm{T}})^{\mathrm{T}} = A_k A_k^{\mathrm{T}}$.
The normed velocity of the kth particle is then

$$\mathbf{\Theta} = \begin{bmatrix} \mathbf{\Theta}_1 \\ \mathbf{\Theta}_2 \\ \vdots \\ \mathbf{\Theta}_N \end{bmatrix} \tag{10a}$$

$$B_k = \begin{bmatrix} l_1 A_1 \\ l_2 A_2 \\ \vdots \\ l_k A_k \\ 0 \end{bmatrix} \tag{10b}$$

$$(\dot{z}_k)^2 = \dot{\mathbf{\Theta}}^{\mathrm{T}} B_k B_k^{\mathrm{T}} \dot{\mathbf{\Theta}}, \tag{10c}$$

where the zero matrix in B_k is a $N-k$ by one zero. Summing over all masses and adding in the potential energy we have for the Lagrangian of the system

$$K = \frac{1}{2}\sum_{k=1}^{N} m_k \dot{\mathbf{\Theta}}^{\mathrm{T}} B_k B_k^{\mathrm{T}} \dot{\mathbf{\Theta}} \tag{11a}$$

$$\mu_k = \sum_{j=k}^{N} m_j \tag{11b}$$

$$\Phi = g\sum_{k=1}^{N} \mu_k l_k \cos\theta_k \tag{11c}$$

$$\mathcal{L} = K - \Phi. \tag{11d}$$

The kinetic energy expressed completely and explicitly as a symmetric quadratic form.

2.2.2. Some tidy up.

Before continuing with evaluation of the Euler-Lagrange equations it is helpful to make a couple of observations about the structure of the matrix products that make up our velocity terms

$$\dot{\mathbf{\Theta}}^{\mathrm{T}} B_k B_k^{\mathrm{T}} \dot{\mathbf{\Theta}} = \dot{\mathbf{\Theta}}^{\mathrm{T}} \begin{bmatrix} \begin{bmatrix} l_1^2 A_1 A_1^{\mathrm{T}} & l_1 l_2 A_1 A_2^{\mathrm{T}} & \dots & l_1 l_k A_1 A_k^{\mathrm{T}} \\ l_2 l_1 A_2 A_1^{\mathrm{T}} & l_2^2 A_2 A_2^{\mathrm{T}} & \dots & l_2 l_k A_2 A_k^{\mathrm{T}} \\ \vdots & & & \\ l_k l_1 A_k A_1^{\mathrm{T}} & l_k l_2 A_k A_2^{\mathrm{T}} & \dots & l_k^2 A_k A_k^{\mathrm{T}} \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \dot{\mathbf{\Theta}}. \tag{12}$$

Pulling in the summation over m_k we have

$$\sum_k m_k \dot{\mathbf{\Theta}}^{\mathrm{T}} B_k B_k^{\mathrm{T}} \dot{\mathbf{\Theta}} = \dot{\mathbf{\Theta}}^{\mathrm{T}} \left[\mu_{\max(r,c)} l_r l_c A_r A_c^{\mathrm{T}} \right]_{rc} \dot{\mathbf{\Theta}}. \tag{13}$$

It appears justifiable to label the $\mu_{\max(r,c)} l_r l_c$ factors of the angular velocity matrices as moments of inertia in a generalized sense. Using this block matrix form, and scalar selection, we can now write the Lagrangian in a slightly tidier form

$$\mu_k = \sum_{j=k}^{N} m_j \tag{14a}$$

$$Q = \left[\mu_{\max(r,c)} l_r l_c A_r A_c^{\mathrm{T}}\right]_{rc} \tag{14b}$$

$$K = \frac{1}{2} \dot{\boldsymbol{\Theta}}^{\mathrm{T}} Q \dot{\boldsymbol{\Theta}} \tag{14c}$$

$$\Phi = g \sum_{k=1}^{N} \mu_k l_k \cos \theta_k \tag{14d}$$

$$\mathcal{L} = K - \Phi. \tag{14e}$$

After some expansion one can find that the block matrices $A_r A_c^{\mathrm{T}}$ contained in Q are

$$A_r A_c^{\mathrm{T}} = \begin{bmatrix} C_{\phi_c - \phi_r} C_{\theta_r} C_{\theta_c} + S_{\theta_r} S_{\theta_c} & -S_{\phi_c - \phi_r} C_{\theta_r} S_{\theta_c} \\ S_{\phi_c - \phi_r} C_{\theta_c} S_{\theta_r} & C_{\phi_c - \phi_r} S_{\theta_r} S_{\theta_c} \end{bmatrix}. \tag{15}$$

The diagonal blocks are particularly simple and have no ϕ dependence

$$A_r A_r^{\mathrm{T}} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2 \theta_r \end{bmatrix} \tag{16}$$

2.2.3. Evaluating the Euler-Lagrange equations.

It will be convenient to group the Euler-Lagrange equations into a column vector form, with a column vector of generalized coordinates and and derivatives, and position and velocity gradients in the associated phase space

$$\mathbf{q} \equiv \left[q_r\right]_r \tag{17}$$

$$\dot{\mathbf{q}} \equiv \left[\dot{q}_r\right]_r \tag{18}$$

$$\nabla_{\mathbf{q}}\mathcal{L} \equiv \left[\frac{\partial \mathcal{L}}{\partial q_r}\right]_r \tag{19}$$

$$\nabla_{\dot{\mathbf{q}}}\mathcal{L} \equiv \left[\frac{\partial \mathcal{L}}{\partial \dot{q}_r}\right]_r. \tag{20}$$

In this form the Euler-Lagrange equations take the form of a single vector equation

$$\nabla_{\mathbf{q}}\mathcal{L} = \frac{d}{dt}\nabla_{\dot{\mathbf{q}}}\mathcal{L}. \tag{21}$$

We are now set to evaluate these generalized phase space gradients. For the acceleration terms our computation reduces nicely to a function of only Q

$$\begin{aligned}
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_a} &= \frac{1}{2}\frac{d}{dt}\left(\frac{\partial \dot{\mathbf{\Theta}}}{\partial \dot{\theta}_a}^{\mathrm{T}} Q\dot{\mathbf{\Theta}} + \dot{\mathbf{\Theta}}^{\mathrm{T}} Q \frac{\partial \dot{\mathbf{\Theta}}}{\partial \dot{\theta}_a}\right) \\
&= \frac{d}{dt}\left(\left[\delta_{ac}\begin{bmatrix}1 & 0\end{bmatrix}\right]_c Q\dot{\mathbf{\Theta}}\right),
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_a} &= \frac{1}{2}\frac{d}{dt}\left(\frac{\partial \dot{\mathbf{\Theta}}}{\partial \dot{\phi}_a}^{\mathrm{T}} Q\dot{\mathbf{\Theta}} + \dot{\mathbf{\Theta}}^{\mathrm{T}} Q \frac{\partial \dot{\mathbf{\Theta}}}{\partial \dot{\phi}_a}\right) \\
&= \frac{d}{dt}\left(\left[\delta_{ac}\begin{bmatrix}0 & 1\end{bmatrix}\right]_c Q\dot{\mathbf{\Theta}}\right).
\end{aligned}$$

The last groupings above made use of $Q = Q^{\mathrm{T}}$, and in particular $(Q + Q^{\mathrm{T}})/2 = Q$. We can now form a column matrix putting all the angular velocity gradient in a tidy block matrix representation

$$\nabla_{\dot{\boldsymbol{\Theta}}}\mathcal{L} = \left[\begin{bmatrix} \frac{\partial \mathcal{L}}{\partial \dot{\theta}_r} \\ \frac{\partial \mathcal{L}}{\partial \dot{\phi}_r} \end{bmatrix} \right]_r = Q\dot{\boldsymbol{\Theta}}. \tag{22}$$

A small aside on Hamiltonian form. This velocity gradient is also the conjugate momentum of the Hamiltonian, so if we wish to express the Hamiltonian in terms of conjugate momenta, we require invertability of Q at the point in time that we evaluate things. Writing

$$P_{\boldsymbol{\Theta}} = \nabla_{\dot{\boldsymbol{\Theta}}}\mathcal{L}, \tag{23}$$

and noting that $(Q^{-1})^{\mathrm{T}} = Q^{-1}$, we get for the kinetic energy portion of the Hamiltonian

$$K = \frac{1}{2} {P_{\boldsymbol{\Theta}}}^{\mathrm{T}} Q^{-1} P_{\boldsymbol{\Theta}}. \tag{24}$$

Now, the invertiblity of Q cannot be taken for granted. Even in the single particle case we do not have invertiblity. For the single particle case we have

$$Q = ml^2 \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix}, \tag{25}$$

so at $\theta = \pm\pi/2$ this quadratic form is singular, and the planar angular momentum becomes a constant of motion.

Returning to the evaluation of the Euler-Lagrange equations, the problem is now reduced to calculating the right hand side of the following system

$$\frac{d}{dt}\left(Q\dot{\boldsymbol{\Theta}}\right) = \left[\begin{bmatrix} \frac{\partial \mathcal{L}}{\partial \theta_r} \\ \frac{\partial \mathcal{L}}{\partial \phi_r} \end{bmatrix} \right]_r. \tag{26}$$

With back substitution of 15, and 16 we have a complete and explicit matrix expansion of the left hand side. For the right hand side taking the θ_a and ϕ_a derivatives respectively we get

$$\frac{\partial \mathcal{L}}{\partial \theta_a} = \frac{1}{2}\dot{\boldsymbol{\Theta}}^{\mathrm{T}} \left[\mu_{\max(r,c)} l_r l_c \left(\frac{\partial A_r}{\partial \theta_a} A_c^{\mathrm{T}} + A_r \frac{\partial A_c}{\partial \theta_a}^{\mathrm{T}} \right) \right]_{rc} \dot{\boldsymbol{\Theta}} - g\mu_a l_a \sin\theta_a, \tag{27a}$$

$$\frac{\partial \mathcal{L}}{\partial \phi_a} = \frac{1}{2}\dot{\boldsymbol{\Theta}}^{\mathrm{T}}\left[\mu_{\max(r,c)} l_r l_c \left(\tfrac{\partial A_r}{\partial \phi_a} A_c^{\mathrm{T}} + A_r \tfrac{\partial A_c}{\partial \phi_a}^{\mathrm{T}}\right)\right]_{rc}\dot{\boldsymbol{\Theta}}. \tag{27b}$$

So to proceed we must consider the $A_r A_c^{\mathrm{T}}$ partials. A bit of thought shows that the matrices of partials above are mostly zeros. Illustrating by example, consider $\partial Q/\partial \theta_2$, which in block matrix form is

$$\frac{\partial Q}{\partial \theta_2} = \begin{bmatrix} 0 & \frac{1}{2}\mu_2 l_1 l_2 A_1 \frac{\partial A_2}{\partial \theta_2}^{\mathrm{T}} & 0 & \dots & 0 \\ \frac{1}{2}\mu_2 l_2 l_1 \frac{\partial A_2}{\partial \theta_2} A_1^{\mathrm{T}} & \frac{1}{2}\mu_2 l_2 l_2 \left(A_2 \frac{\partial A_2}{\partial \theta_2}^{\mathrm{T}} + \frac{\partial A_2}{\partial \theta_2} A_2^{\mathrm{T}} \right) & \frac{1}{2}\mu_3 l_2 l_3 \frac{\partial A_2}{\partial \theta_2} A_3^{\mathrm{T}} & \dots & \frac{1}{2}\mu_N l_2 l_N \frac{\partial A_2}{\partial \theta_2} A_N^{\mathrm{T}} \\ 0 & \frac{1}{2}\mu_3 l_3 l_2 A_3 \frac{\partial A_2}{\partial \theta_2}^{\mathrm{T}} & 0 & \dots & 0 \\ 0 & \vdots & 0 & \dots & 0 \\ 0 & \frac{1}{2}\mu_N l_N l_2 A_N \frac{\partial A_2}{\partial \theta_2}^{\mathrm{T}} & 0 & \dots & 0 \end{bmatrix}. \tag{28}$$

Observe that the diagonal term has a scalar plus its reverse, so we can drop the one half factor and one of the summands for a total contribution to $\partial \mathcal{L}/\partial \theta_2$ of just

$$\mu_2 {l_2}^2 \dot{\boldsymbol{\Theta}}_2^{\mathrm{T}} \frac{\partial A_2}{\partial \theta_2} A_2^{\mathrm{T}} \dot{\boldsymbol{\Theta}}_2.$$

Now consider one of the pairs of off diagonal terms. Adding these we contributions to $\partial \mathcal{L}/\partial \theta_2$ of

$$\begin{aligned} \frac{1}{2}\mu_2 l_1 l_2 \dot{\boldsymbol{\Theta}}_1^{\mathrm{T}} A_1 \frac{\partial A_2}{\partial \theta_2}^{\mathrm{T}} \dot{\boldsymbol{\Theta}}_2 + \frac{1}{2}\mu_2 l_2 l_1 \dot{\boldsymbol{\Theta}}_2^{\mathrm{T}} \frac{\partial A_2}{\partial \theta_2} A_1^{\mathrm{T}} \dot{\boldsymbol{\Theta}}_1 &= \frac{1}{2}\mu_2 l_1 l_2 \dot{\boldsymbol{\Theta}}_1^{\mathrm{T}} A_1 \frac{\partial A_2}{\partial \theta_2}^{\mathrm{T}} + A_1 \frac{\partial A_2}{\partial \theta_2}^{\mathrm{T}} \dot{\boldsymbol{\Theta}}_2 \\ &= \mu_2 l_1 l_2 \dot{\boldsymbol{\Theta}}_1^{\mathrm{T}} A_1 \frac{\partial A_2}{\partial \theta_2}^{\mathrm{T}} \dot{\boldsymbol{\Theta}}_2. \end{aligned}$$

This has exactly the same form as the diagonal term, so summing over all terms we get for the position gradient components of the Euler-Lagrange equation just

$$\frac{\partial \mathcal{L}}{\partial \theta_a} = \sum_k \mu_{\max(k,a)} l_k l_a \dot{\boldsymbol{\Theta}}_k^{\mathrm{T}} A_k \frac{\partial A_a}{\partial \theta_a}^{\mathrm{T}} \dot{\boldsymbol{\Theta}}_a - g \mu_a l_a \sin \theta_a, \tag{29}$$

and

$$\frac{\partial \mathcal{L}}{\partial \phi_a} = \sum_k \mu_{\max(k,a)} l_k l_a \dot{\mathbf{\Theta}}_k^{\mathrm{T}} A_k \frac{\partial A_a}{\partial \phi_a}^{\mathrm{T}} \dot{\mathbf{\Theta}}_a. \tag{30}$$

The only thing that remains to do is evaluate the $A_k \partial A_a / \partial \phi_a{}^{\mathrm{T}}$ matrices.
Utilizing 15, one obtains easily

$$A_k \frac{\partial A_c}{\partial \theta_c}^{\mathrm{T}} = \begin{bmatrix} -C_{\phi_a - \phi_k} C_{\theta_k} S_{\theta_a} + S_{\theta_k} C_{\theta_a} & -S_{\phi_a - \phi_k} C_{\theta_k} C_{\theta_a} \\ -S_{\phi_a - \phi_k} S_{\theta_a} S_{\theta_k} & C_{\phi_a - \phi_k} (1 + \delta_{ka}) S_{\theta_k} C_{\theta_a} \end{bmatrix}, \tag{31}$$

and

$$A_k \frac{\partial A_a}{\partial \phi_a}^{\mathrm{T}} = \begin{bmatrix} -S_{\phi_a - \phi_k} C_{\theta_k} C_{\theta_a} + S_{\theta_k} S_{\theta_a} & -C_{\phi_a - \phi_k} C_{\theta_k} S_{\theta_a} \\ C_{\phi_a - \phi_k} C_{\theta_a} S_{\theta_k} & -S_{\phi_a - \phi_k} S_{\theta_k} S_{\theta_a} \end{bmatrix}. \tag{32}$$

The right hand side of the Euler-Lagrange equations now becomes

$$\nabla_{\mathbf{\Theta}} \mathcal{L} = \sum_k \left[\begin{bmatrix} \mu_{\max(k,r)} l_k l_r \dot{\mathbf{\Theta}}_k^{\mathrm{T}} A_k \frac{\partial A_r}{\partial \theta_r}^{\mathrm{T}} \dot{\mathbf{\Theta}}_r \\ \mu_{\max(k,r)} l_k l_r \dot{\mathbf{\Theta}}_k^{\mathrm{T}} A_k \frac{\partial A_r}{\partial \phi_r}^{\mathrm{T}} \dot{\mathbf{\Theta}}_r \end{bmatrix} \right]_r - g \left[\mu_r l_r \sin\theta_r \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right]_r. \tag{33}$$

Between 33, 22, and a few other auxiliary relations, all above we have completed the task of evaluating the Euler-Lagrange equations for this multiple particle distinct mass system. Unfortunately, just as the simple planar pendulum is a non-linear system, so is this. Possible options for solution are numerical methods or solution restricted to a linear approximation in a small neighborhood of a particular phase space point.

3. **Summary.**

Looking back it is hard to tell the trees from the forest. Here's a summary of the results and definitions of importance. First the Langrangian itself

$$\mu_k = \sum_{j=k}^{N} m_j \tag{34a}$$

$$\boldsymbol{\Theta}_k = \begin{bmatrix} \theta_k \\ \phi_k \end{bmatrix} \tag{34b}$$

$$\boldsymbol{\Theta}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{\Theta}_1^{\mathrm{T}} & \boldsymbol{\Theta}_2^{\mathrm{T}} & \ldots & \boldsymbol{\Theta}_N^{\mathrm{T}} \end{bmatrix} \tag{34c}$$

$$A_k = \begin{bmatrix} C_{\phi_k} C_{\theta_k} & S_{\phi_k} C_{\theta_k} & -S_{\theta_k} \\ -S_{\phi_k} S_{\theta_k} & C_{\phi_k} S_{\theta_k} & 0 \end{bmatrix} \tag{34d}$$

$$Q = \left[\mu_{\max(r,c)} l_r l_c A_r A_c^{\mathrm{T}} \right]_{rc} \tag{34e}$$

$$K = \frac{1}{2} \dot{\boldsymbol{\Theta}}^{\mathrm{T}} Q \dot{\boldsymbol{\Theta}} \tag{34f}$$

$$\Phi = g \sum_{k=1}^{N} \mu_k l_k \cos\theta_k \tag{34g}$$

$$\mathcal{L} = K - \Phi. \tag{34h}$$

Evaluating the Euler-Lagrange equations for the system, we get

$$0 = \nabla_{\boldsymbol{\Theta}} \mathcal{L} - \frac{d}{dt} \left(\nabla_{\dot{\boldsymbol{\Theta}}} \mathcal{L} \right) = \sum_k \left[\begin{bmatrix} \mu_{\max(k,r)} l_k l_r \dot{\boldsymbol{\Theta}}_k^{\mathrm{T}} A_k \frac{\partial A_r}{\partial \theta_r}^{\mathrm{T}} \dot{\boldsymbol{\Theta}}_r \\ \mu_{\max(k,r)} l_k l_r \dot{\boldsymbol{\Theta}}_k^{\mathrm{T}} A_k \frac{\partial A_r}{\partial \phi_r}^{\mathrm{T}} \dot{\boldsymbol{\Theta}}_r \end{bmatrix} \right]_r - g \left[\mu_r l_r \sin\theta_r \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right]_r - \frac{d}{dt} \left(Q \dot{\boldsymbol{\Theta}} \right). \tag{35}$$

Making this explicit requires evaluation of some of the matrix products, and for those we found

$$A_r A_c^{\mathrm{T}} = \begin{bmatrix} C_{\phi_c-\phi_r} C_{\theta_r} C_{\theta_c} + S_{\theta_r} S_{\theta_c} & -S_{\phi_c-\phi_r} C_{\theta_r} S_{\theta_c} \\ S_{\phi_c-\phi_r} C_{\theta_c} S_{\theta_r} & C_{\phi_c-\phi_r} S_{\theta_r} S_{\theta_c} \end{bmatrix} \tag{36a}$$

$$A_k \frac{\partial A_c}{\partial \theta_c}^{\mathrm{T}} = \begin{bmatrix} -C_{\phi_a-\phi_k} C_{\theta_k} S_{\theta_a} + S_{\theta_k} C_{\theta_a} & -S_{\phi_a-\phi_k} C_{\theta_k} C_{\theta_a} \\ -S_{\phi_a-\phi_k} S_{\theta_a} S_{\theta_k} & C_{\phi_a-\phi_k} (1 + \delta_{ka}) S_{\theta_k} C_{\theta_a} \end{bmatrix} \tag{36b}$$

$$A_k \frac{\partial A_a}{\partial \phi_a}^{\mathrm{T}} = \begin{bmatrix} -S_{\phi_a-\phi_k} C_{\theta_k} C_{\theta_a} + S_{\theta_k} S_{\theta_a} & -C_{\phi_a-\phi_k} C_{\theta_k} S_{\theta_a} \\ C_{\phi_a-\phi_k} C_{\theta_a} S_{\theta_k} & -S_{\phi_a-\phi_k} S_{\theta_k} S_{\theta_a} \end{bmatrix}. \tag{36c}$$

4. Conclusions and followup.

This treatment was originally formulated in terms of Geometric Algebra, and matrices of multivector elements were used in the derivation. Being able to compactly specify 3D rotations in a polar form and then factor those vectors into multivector matrix products provides some interesting power, and leads to a structured approach that would perhaps not be obvious otherwise.

In such a formulation the system ends up with a natural Hermitian formulation, where the Hermitian conjugation operations is defined with the vector products reversed, and the matrix elements transposed. Because the vector product is not commutative, some additional care is required in the handling and definition of such matrices, but that is not an insurrmountable problem.

In retrospect it is clear that the same approach is possible with only matrices, and these notes are the result of ripping out all the multivector and Geometric Algebra references in a somewhat brute force fashion. Somewhat sadly, the “pretty” Geometric Algebra methods originally being explored added some complexity to the problem that is not neccessary. It is common to find Geometric Algebra papers and texts show how superior the new non-matrix methods are, and the approach originally used had what was felt to an elegant synthesis of both matrix and GA methods. It is believed that there is still a great deal of potential in such a multivector matrix approach, even if, as in this case, such methods only provide the clarity to understand how to tackle the problem with traditional means.

Because the goals changed in the process of assembling these notes, the reader is justified to complain that this stops prematurely. Lots of math was performed, and then things just end. There ought to be some followup herein to actually do some physics with the end results obtained. Sorry about that.

References

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