Spherical and hyperspherical parameterization.

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Contents

1 Motivation.

In [Joot(a)] a 4D fourier transform solution of Maxwell's equation yielded a Green's function of the form

$$G(x) = \iiiint \frac{e^{ik_\mu x^\mu}}{k_\nu k^\nu} dk_1 dk_2 dk_3 dk_4$$

1

To attempt to "evaluate" this integral, as done in [Joot(b)] to produce the retarded time potentials, a hypervolume equivalent to spherical polar coordinate parameterization is probably desirable.

Before attempting to tackle the problem of interest, the basic question of how to do volume and weighted volume integrals over a hyperspherical volumes must be considered. Doing this for both Euclidean and Minkowski metrics will have to be covered.

2 Euclidean n-volume

2.1 Parameterizations.

The wikipedia article on n-volumes gives a parameterization, which I'll write out explicitly for the first few dimensions

- 1-sphere (circle)

$$x^1 = r \cos \phi_1$$
$$x^2 = r \sin \phi_1$$

- 2-sphere (sphere)

$$x^1 = r \cos \phi_1$$
$$x^2 = r \sin \phi_1 \cos \phi_2$$
$$x^3 = r \sin \phi_1 \sin \phi_2$$

- 3-sphere (hypersphere)

$$x^1 = r \cos \phi_1$$
$$x^2 = r \sin \phi_1 \cos \phi_2$$
$$x^3 = r \sin \phi_1 \sin \phi_2 \cos \phi_3$$
$$x^4 = r \sin \phi_1 \sin \phi_2 \sin \phi_3$$

By inspection one can see that we have the desired $r^2 = \sum_i (x^i)^2$ relation. Each of these can be vectorized to produce a parameterized vector that can trace out all the possible points on the volume

$$\mathbf{r} = \sigma_k x^k$$

2.2 Volume elements.

We can form a parallelogram area (or parallopiped volume, ...) element for any parameterized surface by taking wedge products, as in figure 1. This can also be done for this spherical parameterization too.
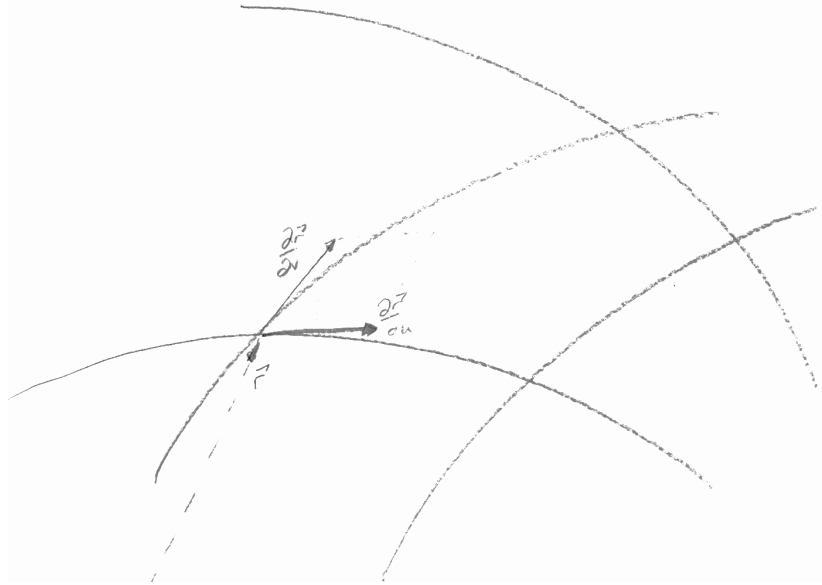

Figure 1: Tangent vector along curves of parameterized vector.

For example for the circle we have

$$
\begin{aligned}
dV_{\mathbb{R}^2} &= \frac{\partial \mathbf{r}}{\partial r} \wedge \frac{\partial \mathbf{r}}{\partial \phi_1} dr d\phi_1 \\
&= \left(\frac{\partial}{\partial r} r (\cos\phi_1, \sin\phi_1) \right) \wedge \left(\frac{\partial}{\partial \phi_1} r (\cos\phi_1, \sin\phi_1) \right) dr d\phi_1 \\
&= (\cos\phi_1, \sin\phi_1) \wedge (-\sin\phi_1, \cos\phi_1) r dr d\phi_1 \\
&= (\cos^2\phi_1 \sigma_1\sigma_2 - \sin^2\phi_1 \sigma_2\sigma_1) r dr d\phi_1 \\
&= r dr d\phi_1 \sigma_1\sigma_2
\end{aligned}
$$

And for the sphere

$$dV_{\mathbb{R}^3} = \frac{\partial \mathbf{r}}{\partial r} \wedge \frac{\partial \mathbf{r}}{\partial \phi_1} \wedge \frac{\partial \mathbf{r}}{\partial \phi_2} dr d\phi_1 d\phi_2$$

$$= (\cos \phi_1, \sin \phi_1 \cos \phi_2, \sin \phi_1 \sin \phi_2)$$

$$\wedge (-\sin \phi_1, \cos \phi_1 \cos \phi_2, \cos \phi_1 \sin \phi_2)$$

$$\wedge (0, -\sin \phi_1 \sin \phi_2, \sin \phi_1 \cos \phi_2) r^2 dr d\phi_1 d\phi_2$$

$$= \begin{vmatrix} \cos \phi_1 & \sin \phi_1 \cos \phi_2 & \sin \phi_1 \sin \phi_2 \\ -\sin \phi_1 & \cos \phi_1 \cos \phi_2 & \cos \phi_1 \sin \phi_2 \\ 0 & -\sin \phi_1 \sin \phi_2 & \sin \phi_1 \cos \phi_2 \end{vmatrix} r^2 dr d\phi_1 d\phi_2 \sigma_1 \sigma_2 \sigma_3$$

$$= r^2 dr \sin \phi_1 d\phi_1 d\phi_2 \sigma_1 \sigma_2 \sigma_3$$

And finally for the hypersphere

$$dV_{\mathbb{R}^4} = \begin{vmatrix} \cos \phi_1 & \sin \phi_1 \cos \phi_2 & \sin \phi_1 \sin \phi_2 \cos \phi_3 & \sin \phi_1 \sin \phi_2 \sin \phi_3 \\ -\sin \phi_1 & \cos \phi_1 \cos \phi_2 & \cos \phi_1 \sin \phi_2 \cos \phi_3 & \cos \phi_1 \sin \phi_2 \sin \phi_3 \\ 0 & -\sin \phi_1 \sin \phi_2 & \sin \phi_1 \cos \phi_2 \cos \phi_3 & \sin \phi_1 \cos \phi_2 \sin \phi_3 \\ 0 & 0 & -\sin \phi_1 \sin \phi_2 \sin \phi_3 & \sin \phi_1 \sin \phi_2 \cos \phi_3 \end{vmatrix}$$

$$r^3 dr d\phi_1 d\phi_2 d\phi_3 \sigma_1 \sigma_2 \sigma_3 \sigma_4$$

$$= r^3 dr \sin^2 \phi_1 d\phi_1 \sin \phi_2 d\phi_2 d\phi_3 \sigma_1 \sigma_2 \sigma_3 \sigma_4$$

Each of these is consistent with the result in the wiki page.

2.3 Some volume computations.

Let's apply the above results to compute the corresponding n-volume's.

- 1-sphere (circle)

$$V_{\mathbb{R}^2} = 4 \int_0^R r dr \int_0^{\pi/2} d\phi_1$$

$$= \pi R^2$$

- 2-sphere (sphere)

$$V_{\mathbb{R}^3} = 8 \int_0^R r^2 dr \int_0^{\pi/2} \sin \phi_1 d\phi_1 \int_0^{\pi/2} d\phi_2$$

$$= 8 \frac{1}{3} R^3 \left(-\cos \phi_1 |_0^{\pi/2} \right) \frac{\pi}{2}$$

$$= \frac{4\pi R^3}{3}$$

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Okay, so far so good.

- 3-sphere (hypersphere)

$$V_{\mathbb{R}^3} = 16 \int_0^R r^3 dr \int_0^{\pi/2} \sin^2 \phi_1 d\phi_1 \int_0^{\pi/2} \sin \phi_2 d\phi_2 \int_0^{\pi/2} d\phi_3$$

$$= 2\pi R^4 \int_0^{\pi/2} \sin^2 \phi_1 d\phi_1$$

$$= \pi R^4 \left(\phi_1 - \cos \phi_1 \sin \phi_1 |_0^{\pi/2} \right)$$

$$= \frac{\pi^2 R^4}{2}$$

This is also consistent with the formula supplied in the wiki article.

2.4 Range determination.

What I've done here though it integrate over only one of the quadrants, and multiply by 2^n. This avoided the more tricky issue of what exact range of angles is required for a complete and non-overlapping cover of the surface.

The wiki article says that the range is $[0, 2\pi]$ for the last angle and $[0, \pi]$ for the others. Reevaluating the integrals above shows that this does work, but that's a bit of a cheat, and it isn't obvious to me past $\mathbb{R}^3$ that this should be the case.

How can this be rationalized?

- circle For the case of the circle what are the end points in each of the quadrants? These are (with $r = 1$)

$$(\cos \phi_1, \sin \phi_1)_{\phi_1=0} = (1, 0) = \sigma_1$$
$$(\cos \phi_1, \sin \phi_1)_{\phi_1=\pi/2} = (0, 1) = \sigma_2$$
$$(\cos \phi_1, \sin \phi_1)_{\phi_1=\pi} = (-1, 0) = -\sigma_1$$
$$(\cos \phi_1, \sin \phi_1)_{\phi_1=3\pi/2} = (0, -1) = -\sigma_2$$

As expected, each of the $\pi/2$ increments traces out the points in successive quadrants.

- sphere

Again with $r = 1$, some representitive points on the circle are

ϕ_1	ϕ_2	$(\cos\phi_1, \sin\phi_1\cos\phi_2, \sin\phi_1\sin\phi_2)$	$\mathbf{r}$
0	0	$(1,0,0)$	σ_1
0	$\pi/2$	$(1,0,0)$	σ_1
0	π	$(1,0,0)$	σ_1
0	$3\pi/2$	$(1,0,0)$	σ_1
$\pi/2$	0	$(0,1,0)$	σ_2
$\pi/2$	$\pi/2$	$(0,0,1)$	σ_3
$\pi/2$	π	$(0,-1,0)$	$-\sigma_2$
$\pi/2$	$3\pi/2$	$(0,0,-1)$	$-\sigma_3$
π	0	$(-1,0,0)$	$-\sigma_1$
π	$\pi/2$	$(-1,0,0)$	$-\sigma_1$
π	π	$(-1,0,0)$	$-\sigma_1$
π	$3\pi/2$	$(-1,0,0)$	$-\sigma_1$

The most informative of these is for $\phi_1 = \pi/2$, where we had $\mathbf{r} = (0, \cos\phi_2, \sin\phi_2)$, and our points trace out a path along the unit circle of the y, z plane. At $\phi_1 = 0$ our point $\mathbf{r} = \sigma_1$ didn't move, and at $\phi_1 = \pi$ we are at the other end of the sphere, also fixed. A reasonable guess is that at each ϕ_1 we trace out a different circle in the y, z plane.

We can write, with $\sigma_{23} = \sigma_1 \wedge \sigma_2 = \sigma_1\sigma_2$,

$$\begin{aligned}
\mathbf{r} &= \cos\phi_1\sigma_1 + \sin\phi_1(\cos\phi_2\sigma_2 + \sin\phi_2\sigma_3) \\
&= \cos\phi_1\sigma_1 + \sin\phi_1\sigma_2(\cos\phi_2 + \sin\phi_2\sigma_2\sigma_3)
\end{aligned}$$

Or, in exponential form

$$\mathbf{r} = \cos\phi_1\sigma_1 + \sin\phi_1\sigma_2\exp(\sigma_{23}\phi_2) \tag{1}$$

Put this way the effects of the parameterization is clear. For each fixed ϕ_1, the exponential traces out a circle in the y, z plane, starting at the point $\mathbf{r} = \cos\phi_1\sigma_1 + \sin\phi_1\sigma_2$. ϕ_1 traces out a semi-circle in the x, y plane.

FIXME: picture.

This would have been easy enough to understand if starting from a picture and constructing the parameterization. Seeing what the geometry is from the algebra requires a bit more (or different) work. Having done it, are we now prepared to understand the geometry of the hypersphere parameterization.

- hypersphere.

  The vector form in the spherical case was convient for extracting geometric properties. Can we do that here too?

$$\begin{aligned}
\mathbf{r} &= \sigma_1 \cos\phi_1 + \sigma_2 \sin\phi_1 \cos\phi_2 + \sigma_3 \sin\phi_1 \sin\phi_2 \cos\phi_3 + \sigma_4 \sin\phi_1 \sin\phi_2 \sin\phi_3 \\
&= \sigma_1 \cos\phi_1 + \sigma_2 \sin\phi_1 \cos\phi_2 + \sigma_3 \sin\phi_1 \sin\phi_2 (\cos\phi_3 + \sigma_{34} \sin\phi_3) \\
&= \sigma_1 \cos\phi_1 + \sigma_2 \sin\phi_1 \cos\phi_2 + \sigma_3 \sin\phi_1 \sin\phi_2 \exp(\sigma_{34}\phi_3) \\
&= \sigma_1 \cos\phi_1 + \sigma_2 \sin\phi_1 (\cos\phi_2 + \sigma_{23} \sin\phi_2 \exp(\sigma_{34}\phi_3))
\end{aligned}$$

Observe that if $\phi_3 = 0$ we have

$$\mathbf{r} = \sigma_1 \cos\phi_1 + \sigma_2 \sin\phi_1 \exp(\sigma_{23}\phi_2)$$

Which is exactly the parameterization of a half sphere ($\phi_2 \in [0, \pi]$). Contrast this to the semi-circle that ϕ_1 traced out in the spherical case.

In the spherical case, the points $\phi_1 = \pi/2$ were nicely representitive. For the hypersphere those points are

$$\mathbf{r} = \sigma_2 \cos\phi_2 + \sigma_3 \sin\phi_2 \exp(\sigma_{34}\phi_3)$$

We saw above that this is the parameterization of a sphere.

Also like the spherical case, we have $\mathbf{r} = \pm\sigma_1$ at $\phi_1 = 0$, and $\phi_1 = \pi$ respectively.

The geometrical conclusion is that for each $\phi_1 \in [0, \pi/2]$ range the points $\mathbf{r}$ trace out increasingly larger spheres, and after that decreasing sized spheres until we get to a point again at $\phi_1 = \pi$.

3 Minkowski metric sphere.

3.1 2D hyperbola.

Our 1-sphere equation was all the points on the curve

$$x^2 + y^2 = r^2$$

The hyperbolic equivalent to this is

$$x^2 - y^2 = r^2$$

Although this is not a closed curve like the circle. To put this in a more natural physical context, lets write (with $c = 1$)

$$\mathbf{r} = \gamma_0 t + \gamma_1 x$$

So the equation of the 1-hyperboliod becomes

$$\mathbf{r}^2 = t^2 - x^2 = r^2$$

We can parameterize this with complex angles $i\phi$

$$\mathbf{r} = r(\gamma_0 \cosh\phi + \gamma_1 \sinh\phi)$$

This gives us

$$\mathbf{r}^2 = r^2(\cosh^2\phi - \sinh^2\phi) = r^2$$

as desired. Like the circle, writing $\gamma_{01} = \gamma_0 \wedge \gamma_1$, an exponential form also works nicely

$$\mathbf{r} = r\gamma_0 \exp(\gamma_{01}\phi)$$

Here the square is

$$
\begin{aligned}
\mathbf{r}^2 &= r^2 \gamma_0 \exp(\gamma_{01}\phi)\gamma_0 \exp(\gamma_{01}\phi) \\
&= r^2 \exp(-\gamma_{01}\phi)(\gamma_0)^2 \exp(\gamma_{01}\phi) \\
&= r^2 \exp(-\gamma_{01}\phi)\exp(\gamma_{01}\phi) \\
&= r^2
\end{aligned}
$$

Again as desired.

3.2   3D hyperbola.

Unlike the circle, a pure hyperbolic parameterization doesn't work to construct a Minkowski square signature. Consider for example

$$\mathbf{r} = \cosh\phi\,\gamma_0 + \gamma_1 \sinh\phi \cosh\psi + \gamma_2 \sinh\phi \sinh\psi$$

Squaring this we have

$$\mathbf{r}^2 = \cosh^2 \phi - \sinh^2 \phi (\cosh^2 \psi + \sinh^2 \psi)$$

We'd get the desired result if we chop off the h in all the ψ hyperbolic functions. This shows that an appropriate parameterization is instead

$$\mathbf{r} = \cosh \phi \gamma_0 + \gamma_1 \sinh \phi \cos \psi + \gamma_2 \sinh \phi \sin \psi$$

This now squares to 1. To see how to extend this to higher dimensions (of which we only need one more) we can factor out a γ_0

$$\mathbf{r} = \gamma_0 (\cosh \phi - \sinh \phi (\underbrace{\sigma_1 \cos \psi + \sigma_2 \sin \psi}_{\text{spatial vector parameterization of circle}}))$$

Now to extend this to three dimensions we have just to substuite the spherical parameterization from 1

$$\mathbf{r} = r\gamma_0 (\cosh \phi_0 - \sinh \phi_0 (\cos \phi_1 \sigma_1 + \sin \phi_1 \sigma_2 \exp(\sigma_{23}\phi_2)))$$
$$= r(\gamma_0 \cosh \phi_0 + \sinh \phi_0 (\cos \phi_1 \gamma_1 + \sin \phi_1 \gamma_2 \exp(\gamma_{32}\phi_2)))$$

3.3 Summarizing the hyperbolic vector parameterizations.

Our parameterization in two, three, and four dimensions, respectively, are

$$\mathbf{r}_2 = r(\gamma_0 \cosh \phi_0 + \sinh \phi_0 \gamma_1)$$
$$\mathbf{r}_3 = r(\gamma_0 \cosh \phi_0 + \sinh \phi_0 \gamma_1 \exp(\gamma_{21}\phi_1))$$
$$\mathbf{r}_4 = r(\gamma_0 \cosh \phi_0 + \sinh \phi_0 (\cos \phi_1 \gamma_1 + \sin \phi_1 \gamma_2 \exp(\gamma_{32}\phi_2)))$$

3.4 Volume elements.

What are our volume elements using this parameterization can be calulated as above.

3.4.1 For one spatial dimension we have

$$dV_2 \gamma_0 \gamma_1 = \begin{vmatrix} \cosh \phi_0 & \sinh \phi_0 \\ \sinh \phi_0 & \cosh \phi_0 \end{vmatrix} r dr d\phi_0$$
$$= r dr d\phi_0$$

3.4.2 For two spatial dimensions we have

$$\mathbf{r}_3 = r(\gamma_0 \cosh \phi_0 + \gamma_1 \sinh \phi_0 \cos \phi_1 + \gamma_2 \sinh \phi_0 \sin \phi_2)$$

The derivatives are

$$\frac{\partial \mathbf{r}_3}{\partial r} = \gamma_0 \cosh \phi_0 + \gamma_1 \sinh \phi_0 \cos \phi_1 + \gamma_2 \sinh \phi_0 \sin \phi_1$$

$$\frac{1}{r}\frac{\partial \mathbf{r}_3}{\partial \phi_0} = \gamma_0 \sinh \phi_0 + \gamma_1 \cosh \phi_0 \cos \phi_1 + \gamma_2 \cosh \phi_0 \sin \phi_1$$

$$\frac{1}{r}\frac{\partial \mathbf{r}_3}{\partial \phi_1} = -\gamma_1 \sinh \phi_0 \sin \phi_1 + \gamma_2 \sinh \phi_0 \cos \phi_1$$

Or

$$\frac{\partial \mathbf{r}_3}{\partial r} = \gamma_0 \cosh \phi_0 + \gamma_1 \sinh \phi_0 \exp(\gamma_{21}\phi_1)$$

$$\frac{1}{r}\frac{\partial \mathbf{r}_3}{\partial \phi_0} = \gamma_0 \sinh \phi_0 + \cosh \phi_0 \exp(-\gamma_{21}\phi_1)\gamma_1$$

$$\frac{1}{r}\frac{\partial \mathbf{r}_3}{\partial \phi_1} = \sinh \phi_0 \gamma_2 \exp(\gamma_{21}\phi_1)$$

Multiplying this out, discarding non-grade three terms we have

$$(\gamma_{10} \sinh^2 \phi_0 \exp(\gamma_{21}\phi_1) + \gamma_{01} \cosh^2 \phi_0 \exp(\gamma_{21}\phi_1)) \sinh \phi_0 \gamma_2 \exp(\gamma_{21}\phi_1)$$
$$= \gamma_{01} \exp(\gamma_{21}\phi_1) \sinh \phi_0 \exp(-\gamma_{21}\phi_1)\gamma_2$$
$$= \gamma_{01} \sinh \phi_0 \gamma_2$$

This gives us

$$dV_3 = r^2 \sinh \phi_0 dr d\phi_0 d\phi_1$$

3.4.3 For three spatial dimensions we have

$$\mathbf{r}_4 = r(\gamma_0 \cosh \phi_0 + \sinh \phi_0 (\cos \phi_1 \gamma_1 + \sin \phi_1 \gamma_2 \exp(\gamma_{32}\phi_2)))$$

So our derivatives are

$$\frac{\partial \mathbf{r}_4}{\partial r} = \gamma_0 \cosh \phi_0 + \sinh \phi_0 (\cos \phi_1 \gamma_1 + \sin \phi_1 \gamma_2 \exp(\gamma_{32} \phi_2))$$

$$\frac{1}{r} \frac{\partial \mathbf{r}_4}{\partial \phi_0} = \gamma_0 \sinh \phi_0 + \cosh \phi_0 (\cos \phi_1 \gamma_1 + \sin \phi_1 \gamma_2 \exp(\gamma_{32} \phi_2))$$

$$\frac{1}{r} \frac{\partial \mathbf{r}_4}{\partial \phi_1} = \sinh \phi_0 (-\sin \phi_1 \gamma_1 + \cos \phi_1 \gamma_2 \exp(\gamma_{32} \phi_2))$$

$$\frac{1}{r} \frac{\partial \mathbf{r}_4}{\partial \phi_2} = \sinh \phi_0 \sin \phi_1 \gamma_3 \exp(\gamma_{32} \phi_2)$$

In shorthand, writing C and S for the trig and hyperbolic functions as appropriate, we have

$$\gamma_0 C_0 + S_0 C_1 \gamma_1 + S_0 S_1 \gamma_2 \exp(\gamma_{32} \phi_2)$$
$$\gamma_0 S_0 + C_0 C_1 \gamma_1 + C_0 S_1 \gamma_2 \exp(\gamma_{32} \phi_2)$$
$$-S_0 S_1 \gamma_1 + S_0 C_1 \gamma_2 \exp(\gamma_{32} \phi_2)$$
$$S_0 S_1 \gamma_3 \exp(\gamma_{32} \phi_2)$$

Multiplying these out and dropping terms that will not contribute grade four bits is needed to calculate the volume element. The full product for the first two derivatives is

$$\gamma_0 C_0 \gamma_0 S_0 + \gamma_0 C_0 C_0 C_1 \gamma_1 + \gamma_0 C_0 C_0 S_1 \gamma_2 \exp(\gamma_{32} \phi_2)$$
$$+ S_0 C_1 \gamma_1 \gamma_0 S_0 + S_0 C_1 \gamma_1 C_0 C_1 \gamma_1 + S_0 C_1 \gamma_1 C_0 S_1 \gamma_2 \exp(\gamma_{32} \phi_2)$$
$$+ S_0 S_1 \gamma_2 \exp(\gamma_{32} \phi_2) \gamma_0 S_0 + S_0 S_1 \gamma_2 \exp(\gamma_{32} \phi_2) C_0 C_1 \gamma_1 + S_0 S_1 \gamma_2 \exp(\gamma_{32} \phi_2) C_0 S_1 \gamma_2 \exp(\gamma_{32} \phi_2)$$

We can discard the scalar terms:

$$\gamma_0 C_0 \gamma_0 S_0 + S_0 C_1 \gamma_1 C_0 C_1 \gamma_1 + S_0 S_1 \gamma_2 \exp(\gamma_{32} \phi_2) C_0 S_1 \exp(-\gamma_{32} \phi_2) \gamma_2$$

Two of these terms cancel out

$$S_0 C_0 C_1 S_1 \gamma_{12} \exp(\gamma_{32} \phi_2) + S_0 C_0 C_1 S_1 \gamma_{21} \exp(\gamma_{32} \phi_2)$$

and we are left with two bivector contributors to the eventual four-pseudoscalar

$$\gamma_{01} C_1 + \gamma_{02} S_1 \exp(\gamma_{32} \phi_2)$$

Multiplying out the last two derivatives we have

$$-S_0^2 S_1^2 \gamma_{13} \exp(\gamma_{32} \phi_2) + S_0^2 C_1 S_1 \gamma_{23}$$

Almost there. A final multiplication of these sets of products gives

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$$- \gamma_{01} C_1 S_0^2 S_1^2 \gamma_{13} \exp(\gamma_{32}\phi_2) - \gamma_{02} S_1 \exp(\gamma_{32}\phi_2) S_0^2 S_1^2 \gamma_{13} \exp(\gamma_{32}\phi_2)$$
$$+ \gamma_{01} C_1 S_0^2 C_1 S_1 \gamma_{23} + \gamma_{02} S_1 \exp(\gamma_{32}\phi_2) S_0^2 C_1 S_1 \gamma_{23}$$
$$= \gamma_{03} C_1 S_0^2 S_1^2 \exp(\gamma_{32}\phi_2) - \gamma_{02} \exp(\gamma_{32}\phi_2) S_0^2 S_1^3 \exp(-\gamma_{32}\phi_2) \gamma_{13} + \gamma_{0123} C_1^2 S_0^2 S_1 - \gamma_{03} \exp(\gamma_{32}\phi_2) S_0^2 C_1 S_1^2$$
$$= \gamma_{03} C_1 S_0^2 S_1^2 \exp(\gamma_{32}\phi_2) + \gamma_{0123} S_0^2 S_1 (S_1^2 + C_1^2) - \gamma_{03} \exp(\gamma_{32}\phi_2) S_0^2 C_1 S_1^2$$
$$= \gamma_{03} C_1 S_1^2 (S_0^2 - S_0^2) \exp(\gamma_{32}\phi_2) + \gamma_{0123} S_0^2 S_1$$

Therefore our final result is

$$dV_4 = \sinh^2 \phi_0 \sin \phi_1 r^3 dr d\phi_0 d\phi_1 d\phi_2$$

4   Summary.

4.1   Vector parameterizations.

N-Spherical parameterization

$$\mathbf{r}_2 = \sigma_1 \exp(\sigma_{12}\phi_1) \tag{2}$$
$$\mathbf{r}_3 = \cos\phi_1 \sigma_1 + \sin\phi_1 \sigma_2 \exp(\sigma_{23}\phi_2) \tag{3}$$
$$\mathbf{r}_4 = \sigma_1 \cos\phi_1 + \sigma_2 \sin\phi_1 (\cos\phi_2 + \sigma_{23} \sin\phi_2 \exp(\sigma_{34}\phi_3)) \tag{4}$$

N-Hypersphere parameterization

$$\mathbf{r}_2 = r(\gamma_0 \cosh\phi_0 + \sinh\phi_0 \gamma_1) \tag{5}$$
$$\mathbf{r}_3 = r(\gamma_0 \cosh\phi_0 + \sinh\phi_0 \gamma_1 \exp(\gamma_{21}\phi_1)) \tag{6}$$
$$\mathbf{r}_4 = r(\gamma_0 \cosh\phi_0 + \sinh\phi_0 (\cos\phi_1 \gamma_1 + \sin\phi_1 \gamma_2 \exp(\gamma_{32}\phi_2))) \tag{7}$$

4.2   Volume elements.

To summarize the mess of algebra we have shown that our hyperbolic volume elements are given by

$$dV_2 = (r dr) \, d\phi_0 \tag{8}$$
$$dV_3 = \left(r^2 dr\right) (\sinh\phi_0 d\phi_0) \, d\phi_1 \tag{9}$$
$$dV_4 = \left(r^3 dr\right) \left(\sinh^2 \phi_0 d\phi_0\right) (\sin\phi_1 d\phi_1) \, d\phi_2 \tag{10}$$

Compare this to the volume elements for the n-spheres

$$dV_2 = (r\,dr)\,d\phi_1 \tag{11}$$

$$dV_3 = \left(r^2 dr\right)(\sin\phi_1 d\phi_1)\,d\phi_2 \tag{12}$$

$$dV_4 = \left(r^3 dr\right)\left(\sin^2\phi_1 d\phi_1\right)(\sin\phi_2 d\phi_2)\,d\phi_3 \tag{13}$$

Besides labling variations the only difference in the form is a switch from trig to hyperbolic functions for the first angle (which has an implied range difference as well).

References

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[Joot(b)] Peeter Joot. Poisson and retarded potential green's functions from fourier kernels. "`http://sites.google.com/site/peeterjoot/math2009/poisson.pdf`", b.