Lorentz force relation to the energy momentum tensor.

Peeter Joot

Feb 13, 2009. Last Revision: $Date : 2009/02/1504 : 53 : 32$

Contents

1 Motivation. 1

2 Spacetime divergence of the energy momentum four vectors. 3

3 Four vector Lorentz Force. 4
- 3.1 Lorentz force in tensor form. 4
- 3.2 Lorentz force components in vector form. 5
- 3.3 Relation to the energy momentum tensor. 5

4 Expansion of the energy momentum tensor. 6
- 4.1 In terms of electric and magnetic field components. 6
  - 4.1.1 First row. 6
  - 4.1.2 First column. 7
  - 4.1.3 Diagonal and remaining terms. 7
- 4.2 Summarizing. 9
- 4.3 Assemblying a four vector. 9

5 Conjugation? 10
- 5.1 Followup: energy momentum tensor. 10

1 Motivation.

Have now made a few excursions related to the concepts of electrodynamic field energy and momentum.

In [Joot(e)] the energy density rate and poynting divergence relationship was demonstrated using Maxwell's equation. That was:

$$\frac{\partial}{\partial t}\frac{\epsilon_0}{2}\left(\mathbf{E}^2 + c^2\mathbf{B}^2\right) + \boldsymbol{\nabla}\cdot\frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = -\mathbf{E}\cdot\mathbf{j} \tag{1}$$

In terms of the field energy density U, and Poynting vector $\mathbf{P}$, this is

$$U = \frac{\epsilon_0}{2}\left(\mathbf{E}^2 + c^2\mathbf{B}^2\right) \tag{2}$$

$$\mathbf{P} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) \tag{3}$$

$$\frac{\partial U}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{P} = -\mathbf{E} \cdot \mathbf{j} \tag{4}$$

In [Joot(c)] this was related to the energy momentum four vectors

$$T(a) = \frac{\epsilon_0}{2} F a \tilde{F} \tag{5}$$

as defined in [Doran and Lasenby(2003)], but the big picture view of things was missing.

Later in [Joot(f)] the rate of change of Poynting vector was calculated, with an additional attempt to relate this to $T(\gamma_\mu)$.

These relationships, and the operations required to factoring out the divergence were considerably messier.

Finally, in [Joot(b)] the four vector $T(\gamma_\mu)$ was related to the Lorentz force and the work done moving a charge against a field. This provides the natural context for the energy momentum tensor, since it appears that the spacetime divergence of each of the $T(\gamma_\mu)$ four vectors appears to be a component of the four vector Lorentz force (density).

In these notes the divergences will be calculated to confirm the connection between the Lorentz force and energy momentum tensor directly. This is actually expected to be simpler than the previous calculations.

It is also potentially of interest, as shown in [Joot(a)], and [Joot(d)] that the energy density and Poynting vectors, and energy momentum four vector, were seen to be naturally expressable as Hermitian conjugate operations

$$F^\dagger = \gamma_0 \tilde{F} \gamma_0 \tag{6}$$

$$T(\gamma_0) = \frac{\epsilon_0}{2} F F^\dagger \gamma_0 \tag{7}$$

$$U = T(\gamma_0) \cdot \gamma_0 = \frac{\epsilon_0}{4}\left(F F^\dagger + F^\dagger F\right) \tag{8}$$

$$\mathbf{P}/c = T(\gamma_0) \wedge \gamma_0 = \frac{\epsilon_0}{4}\left(F F^\dagger - F^\dagger F\right) \tag{9}$$

It is concievable that a generalization of Hermitian conjugation, where the spatial basis vectors are used instead of γ_0, will provide a mapping and driving structure from the Four vector quantities and the somewhat scrambled seeming set of relationships observed in the split spatial and time domain. That will also be explored here.

2 Spacetime divergence of the energy momentum four vectors.

The spacetime divergence of the field energy momentum four vector $T(\gamma_0)$ has been calculated previously. Let's redo this calculation for the other components.

$$
\begin{aligned}
\nabla \cdot T(\gamma_\mu) &= \frac{\epsilon_0}{2}\left\langle \nabla(F\gamma_\mu\tilde{F})\right\rangle \\
&= \frac{\epsilon_0}{2}\left\langle (\nabla F)\gamma_\mu\tilde{F} + (\tilde{F}\nabla)F\gamma_\mu\right\rangle \\
&= \frac{\epsilon_0}{2}\left\langle (\nabla F)\gamma_\mu\tilde{F} + \gamma_\mu\tilde{F}(\nabla F)\right\rangle \\
&= \epsilon_0\left\langle (\nabla F)\gamma_\mu\tilde{F}\right\rangle \\
&= \frac{1}{c}\left\langle J\gamma_\mu\tilde{F}\right\rangle
\end{aligned}
$$

The ability to perform cyclic reordering of terms in a scalar product has been used above. Application of one more reverse operation (which doesn't change a scalar), gives us

$$
\nabla \cdot T(\gamma_\mu) = \frac{1}{c}\left\langle F\gamma_\mu J\right\rangle \tag{10}
$$

Let's expand the right hand size first.

$$
\frac{1}{c}\left\langle F\gamma_\mu J\right\rangle = \frac{1}{c}\left\langle (\mathbf{E} + ic\mathbf{B})\gamma_\mu(c\rho\gamma_0 + \mathbf{j}\gamma_0)\right\rangle
$$

The $\mu = 0$ term looks the easiest, and for that one we have

$$
\frac{1}{c}\langle(\mathbf{E} + ic\mathbf{B})(c\rho - \mathbf{j})\rangle = -\mathbf{j}\cdot\mathbf{E}
$$

Now, for the other terms, say $\mu = k$, we have

$$
\begin{aligned}
\frac{1}{c}\langle(\mathbf{E} + ic\mathbf{B})(c\rho\sigma_k - \sigma_k\mathbf{j})\rangle &= E^k\rho - \langle i\mathbf{B}\sigma_k\mathbf{j}\rangle \\
&= E^k\rho - J^a B^b\langle\sigma_1\sigma_2\sigma_3\sigma_b\sigma_k\sigma_a\rangle \\
&= E^k\rho - J^a B^b\epsilon_{akb} \\
&= E^k\rho + J^a B^b\epsilon_{kab} \\
&= (\rho\mathbf{E} + \mathbf{j}\times\mathbf{B})\cdot\sigma_k
\end{aligned}
$$

Summarizing the two results we have

$$\frac{1}{c}\langle F\gamma_0 J\rangle = -\mathbf{j}\cdot\mathbf{E} \tag{11}$$

$$\frac{1}{c}\langle F\gamma_k J\rangle = (\rho\mathbf{E} + \mathbf{j}\times\mathbf{B})\cdot\sigma_k \tag{12}$$

The second of these is easily recognizable as components of the Lorentz force for an element of charge (density). The first of these is actually the energy component of the four vector Lorentz force, so expanding that in terms of spacetime quantities is the next order of business.

3 Four vector Lorentz Force.

The Lorentz force in covariant form is

$$m\ddot{x} = qF\cdot\dot{x}/c \tag{13}$$

Two verifications of this are in order. One is that we get the traditional vector form of the Lorentz force equation from this and the other is that we can get the traditional tensor form from this equation.

3.1 Lorentz force in tensor form.

Recovering the tensor form is probably the easier of the two operations. We have

$$\begin{aligned}
m\ddot{x}_\mu\gamma^\mu &= \frac{q}{2}F_{\alpha\beta}\dot{x}_\sigma(\gamma^\alpha\wedge\gamma^\beta)\cdot\gamma^\sigma \\
&= \frac{q}{2}F_{\alpha\beta}\dot{x}^\sigma(\gamma^\alpha{\delta^\beta}_\sigma - \gamma^\beta{\delta^\alpha}_\sigma) \\
&= \frac{q}{2}F_{\alpha\beta}\dot{x}^\beta\gamma^\alpha - \frac{q}{2}F_{\alpha\beta}\dot{x}^\alpha\gamma^\beta
\end{aligned}$$

Dotting with γ_μ the right hand side is

$$\frac{q}{2}F_{\mu\beta}\dot{x}^\beta - \frac{q}{2}F_{\alpha\mu}\dot{x}^\alpha = qF_{\mu\alpha}\dot{x}^\alpha$$

Which recovers the tensor form of the equation

$$m\ddot{x}_\mu = qF_{\mu\alpha}\dot{x}^\alpha \tag{14}$$

3.2 Lorentz force components in vector form.

$$
\begin{aligned}
m\gamma \frac{d}{dt}\gamma \left(c + \sigma_k \frac{dx^k}{dt} \right) \gamma_0 &= \frac{q}{2c}(Fv - vF) \\
&= \frac{q\gamma}{2c}(\mathbf{E} + ic\mathbf{B}) \left(c + \sigma_k \frac{dx^k}{dt} \right) \gamma_0 \\
&\quad - \frac{q\gamma}{2c} \left(c + \sigma_k \frac{dx^k}{dt} \right) \gamma_0 (\mathbf{E} + ic\mathbf{B})
\end{aligned}
$$

Right multiplication by γ_0/γ we have

$$
\begin{aligned}
m\frac{d}{dt}\gamma\left(c + \mathbf{v}\right) &= \frac{q}{2c}\left(\left(\mathbf{E} + ic\mathbf{B}\right)\left(c + \mathbf{v}\right) - \left(c + \mathbf{v}\right)\left(-\mathbf{E} + ic\mathbf{B}\right)\right) \\
&= \frac{q}{2c}\left(+2\mathbf{E}c + \mathbf{E}\mathbf{v} + \mathbf{v}\mathbf{E} + ic(\mathbf{B}\mathbf{v} - \mathbf{v}\mathbf{B})\right)
\end{aligned}
$$

After a last bit of reduction this is

$$
m\frac{d}{dt}\gamma\left(c + \mathbf{v}\right) = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + q\mathbf{E} \cdot \mathbf{v}/c \tag{15}
$$

In terms of four vector momentum this is

$$
\dot{p} = q(\mathbf{E} \cdot \mathbf{v}/c + \mathbf{E} + \mathbf{v} \times \mathbf{B})\gamma_0 \tag{16}
$$

3.3 Relation to the energy momentum tensor.

It appears that to relate the energy momentum tensor to the Lorentz force we have to work with the upper index quantities rather than the lower index stress tensor vectors. Doing so our four vector force per unit volume is

$$
\begin{aligned}
\frac{\partial \dot{p}}{\partial V} &= (\mathbf{j} \cdot \mathbf{E} + \rho\mathbf{E} + \mathbf{j} \times \mathbf{B})\gamma_0 && (17) \\
&= -\frac{1}{c}\langle F\gamma^\mu J \rangle \gamma_\mu && (18) \\
&= -(\nabla \cdot T(\gamma^\mu))\gamma_\mu && (19)
\end{aligned}
$$

The term $\langle F\gamma^\mu J \rangle \gamma_\mu$ appears to be expressed simply has $F \cdot J$ in [Doran and Lasenby(2003)]. Understanding that simple statement is now possible now that an exploration of some of the underlying ideas has been made. In retrospect having seen the bivector product form of the Lorentz force equation, it should have been clear, but some of the associated trickiness in their treatment obscured this fact (Although their treatment is only two pages, I still only understand half of what they are doing!)

4 Expansion of the energy momentum tensor.

While all the components of the divergence of the energy momentum tensor have been expanded explicitly, this hasn't been done here for the tensor itself. A mechanical expansion of the tensor in terms of field tensor components $F^{\mu\nu}$ has been done previously and isn't particularily enlightening. Let's work it out here in terms of electric and magnetic field components. In particular for the $T^{0\mu}$ and $T^{\mu 0}$ components of the tensor in terms of energy density and the Poynting vector.

4.1 In terms of electric and magnetic field components.

Here we want to expand

$$T(\gamma^\mu) = \frac{-\epsilon_0}{2}(\mathbf{E} + ic\mathbf{B})\gamma^\mu(\mathbf{E} + ic\mathbf{B})$$

It will be convienient here to temporarily work with $\epsilon_0 = c = 1$, and put them back in afterwards.

4.1.1 First row.

First expanding $T(\gamma^0)$ we have

$$\begin{aligned}
T(\gamma^0) &= \frac{1}{2}(\mathbf{E} + i\mathbf{B})(\mathbf{E} - i\mathbf{B})\gamma^0 \\
&= \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2 + i(\mathbf{B}\mathbf{E} - \mathbf{E}\mathbf{B}))\gamma^0 \\
&= \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)\gamma^0 + i(\mathbf{B} \wedge \mathbf{E})\gamma^0
\end{aligned}$$

Using the wedge product dual $\mathbf{a} \wedge \mathbf{b} = i(\mathbf{a} \times \mathbf{b})$, and putting back in the units, we have our first stress energy four vector,

$$T(\gamma^0) = \left(\frac{\epsilon_0}{2}(\mathbf{E}^2 + c^2\mathbf{B}^2) + \frac{1}{\mu_0 c}(\mathbf{E} \times \mathbf{B})\right)\gamma^0 \tag{20}$$

In particular the energy density and the components of the Poynting vector can be picked off by dotting with each of the γ^μ vectors. That is

$$U = T(\gamma^0) \cdot \gamma^0 \tag{21}$$

$$\mathbf{P}/c \cdot \sigma_k = T(\gamma^0) \cdot \gamma^k \tag{22}$$

4.1.2 First column.

We have Poynting vector terms in the T^{0k} elements of the matrix. Let's quickly verify that we have them in the T^{k0} positions too.

To do so, again with $c = \epsilon_0 = 1$ temporarily this is a computation of

$$
\begin{aligned}
T(\gamma^k) \cdot \gamma^0 &= \frac{1}{2}(T(\gamma^k)\gamma^0 + \gamma^0 T(\gamma^k)) \\
&= \frac{-1}{4}(F\gamma^k F\gamma^0 + \gamma^0 F\gamma^k F) \\
&= \frac{1}{4}(F\sigma_k\gamma_0 F\gamma^0 - \gamma^0 F\gamma_0\sigma_k F) \\
&= \frac{1}{4}(F\sigma_k(-\mathbf{E} + i\mathbf{B}) - (-\mathbf{E} + i\mathbf{B})\sigma_k F) \\
&= \frac{1}{4}\langle \sigma_k(-\mathbf{E} + i\mathbf{B})(\mathbf{E} + i\mathbf{B}) - \sigma_k(\mathbf{E} + i\mathbf{B})(-\mathbf{E} + i\mathbf{B})\rangle \\
&= \frac{1}{4}\Big\langle \sigma_k(-\mathbf{E}^2 - \mathbf{B}^2 + 2(\mathbf{E} \times \mathbf{B})) - \sigma_k(-\mathbf{E}^2 - \mathbf{B}^2 - 2(\mathbf{E} \times \mathbf{B}))\Big\rangle
\end{aligned}
$$

Adding back in the units we have

$$
T(\gamma^k) \cdot \gamma^0 = \epsilon_0 c(\mathbf{E} \times \mathbf{B}) \cdot \sigma_k = \frac{1}{c}\mathbf{P} \cdot \sigma_k \tag{23}
$$

As expected, these are the components of the Poynting vector (scaled by $1/c$ for units of energy density).

4.1.3 Diagonal and remaining terms.

$$
\begin{aligned}
T(\gamma^a) \cdot \gamma^b &= \frac{1}{2}(T(\gamma^a)\gamma^b + \gamma^b T(\gamma^a)) \\
&= \frac{-1}{4}(F\gamma^a F\gamma^b + \gamma^a F\gamma^b F) \\
&= \frac{1}{4}(F\sigma_a\gamma_0 F\gamma^b - \gamma^a F\gamma_0\sigma_b F) \\
&= \frac{1}{4}(F\sigma_a(-\mathbf{E} + i\mathbf{B})\sigma_b + \sigma_a(-\mathbf{E} + i\mathbf{B})\sigma_b F) \\
&= \frac{1}{2}\langle \sigma_a(-\mathbf{E} + i\mathbf{B})\sigma_b(\mathbf{E} + i\mathbf{B})\rangle
\end{aligned}
$$

From this point is there any particularily good or clever way to do the remaining reduction? Doing it with coordinates looks like it would be easy, but also messy. A decomposition of **E** and **B** that are parallel and perpendicular to the spatial basis vectors also looks feasable.

Let's try the dumb way first

$$
\begin{aligned}
T(\gamma^a) \cdot \gamma^b &= \frac{1}{2} \left\langle \sigma_a (-E^k \sigma_k + i B^k \sigma_k) \sigma_b (E^m \sigma_m + i B^m \sigma_m) \right\rangle \\
&= \frac{1}{2} (B^k E^m - E^k B^m) \langle i \sigma_a \sigma_k \sigma_b \sigma_m \rangle - \frac{1}{2} (E^k E^m + B^k B^m) \langle \sigma_a \sigma_k \sigma_b \sigma_m \rangle
\end{aligned}
$$

Reducing the scalar operations is going to be much different for the $a = b$, and $a \neq b$ cases. For the diagonal case we have

$$
\begin{aligned}
T(\gamma^a) \cdot \gamma^a &= \frac{1}{2} (B^k E^m - E^k B^m) \langle i \sigma_a \sigma_k \sigma_a \sigma_m \rangle - \frac{1}{2} (E^k E^m + B^k B^m) \langle \sigma_a \sigma_k \sigma_a \sigma_m \rangle \\
&= -\frac{1}{2} \sum_{m,k \neq a} \frac{1}{2} (B^k E^m - E^k B^m) \langle i \sigma_k \sigma_m \rangle + \frac{1}{2} \sum_{m,k \neq a} (E^k E^m + B^k B^m) \langle \sigma_k \sigma_m \rangle \\
&+ \frac{1}{2} \sum_m (B^a E^m - E^a B^m) \langle i \sigma_a \sigma_m \rangle - \frac{1}{2} \sum_m (E^a E^m + B^a B^m) \langle \sigma_a \sigma_m \rangle
\end{aligned}
$$

Inserting the units again we have

$$
T(\gamma^a) \cdot \gamma^a = \frac{\epsilon_0}{2} \left(\sum_{k \neq a} \left((E^k)^2 + c^2 (B^k)^2 \right) - \left((E^a)^2 + c^2 (B^a)^2 \right) \right) \tag{24}
$$

Or, adding and subtracting, we have the diagonal in terms of energy density (minus a fudge)

$$
T(\gamma^a) \cdot \gamma^a = U - \epsilon_0 \left((E^a)^2 + c^2 (B^a)^2 \right) \tag{25}
$$

Now, for the off diagonal terms. For $a \neq b$ this is

$$
\begin{aligned}
T(\gamma^a) \cdot \gamma^b &= \frac{1}{2} \sum_m (B^a E^m - E^a B^m) \langle i \sigma_b \sigma_m \rangle + \frac{1}{2} \sum_m (B^b E^m - E^b B^m) \langle i \sigma_a \sigma_m \rangle \\
&- \frac{1}{2} \sum_m (E^a E^m + B^a B^m) \langle \sigma_b \sigma_m \rangle - \frac{1}{2} \sum_m (E^b E^m + B^b B^m) \langle \sigma_a \sigma_m \rangle \\
&+ \frac{1}{2} \sum_{m,k \neq a,b} (B^k E^m - E^k B^m) \langle i \sigma_a \sigma_k \sigma_b \sigma_m \rangle - \frac{1}{2} \sum_{m,k \neq a,b} (E^k E^m + B^k B^m) \langle \sigma_a \sigma_k \sigma_b \sigma_m \rangle
\end{aligned}
$$

The first two scalar filters that include i will be zero, and we have deltas $\langle \sigma_b \sigma_m \rangle = \delta_{bm}$ in the next two. The remaining two terms have only vector and bivector terms, so we have zero scalar parts. That leaves (restoring units)

$$T(\gamma^a) \cdot \gamma^b = -\frac{\epsilon_0}{2} \left(E^a E^b + E^b E^a + c^2 (B^a B^b + B^b B^a) \right) \tag{26}$$

4.2 Summarizing.

Collecting all the results, with $T^{\mu\nu} = T(\gamma^\mu) \cdot \gamma^\nu$, we have

$$\begin{aligned}
T^{00} &= \frac{\epsilon_0}{2} \left(\mathbf{E}^2 + c^2 \mathbf{B}^2 \right) && (27) \\
T^{aa} &= \frac{\epsilon_0}{2} \left(\mathbf{E}^2 + c^2 \mathbf{B}^2 \right) - \epsilon_0 \left((E^a)^2 + c^2 (B^a)^2 \right) && (28) \\
T^{k0} = T^{0k} &= \frac{1}{c} \left(\frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B}) \right) \cdot \sigma_k && (29) \\
T^{ab} = T^{ba} &= -\frac{\epsilon_0}{2} \left(E^a E^b + E^b E^a + c^2 (B^a B^b + B^b B^a) \right) && (30)
\end{aligned}$$

4.3 Assemblying a four vector.

Let's see what one of the $T^{a\mu}\gamma_\mu$ rows of the tensor looks like in four vector form. Let $f \neq g \neq h$ represent an even permutation of the integers $1, 2, 3$. Then we have

$$\begin{aligned}
T^f &= T^{f\mu}\gamma_\mu \\
&= \frac{\epsilon_0}{2} c (E^g B^h - E^h B^g) \gamma_0 \\
&+ \frac{\epsilon_0}{2} \left(-(E^f)^2 + (E^g)^2 + (E^h)^2 + c^2 (-(B^f)^2 + (B^g)^2 + (B^h)^2) \right) \gamma_f \\
&- \frac{\epsilon_0}{2} \left(E^f E^g + E^g E^f + c^2 (B^f B^g + B^g B^f) \right) \gamma_g \\
&- \frac{\epsilon_0}{2} \left(E^f E^h + E^h E^f + c^2 (B^f B^h + B^h B^f) \right) \gamma_h
\end{aligned}$$

It is pretty amazing that the divergence of this produces the f component of the Lorentz force (density)

$$\partial_\mu T^{f\mu} = (\rho \mathbf{E} + \mathbf{j} \times \mathbf{B}) \cdot \sigma_f \tag{31}$$

Demonstrating this directly without having STA as an available tool would be quite tedious, and looking at this expression inspires no particular attempt to try!

5 Conjugation?

5.1 Followup: energy momentum tensor.

This also suggests a relativistic generalization of conjugation, since the time basis vector should perhaps not have a distinguishing role. Something like this:

$$F^{\dagger_\mu} = \gamma_\mu \tilde{F} \gamma_\mu$$

Or perhaps:

$$F^{\dagger_\mu} = \gamma_\mu \tilde{F} \gamma^\mu$$

may make sense for consideration of the other components of the general energy momentum tensor, which had roughly the form:

$$T^{\mu\nu} \propto T(\gamma_\mu) \cdot \gamma^\nu$$

(with some probable adjustments to index positions). Think this through later.

References

[Doran and Lasenby(2003)] C. Doran and A.N. Lasenby. *Geometric algebra for physicists*. Cambridge University Press New York, 2003.

[Joot(a)] Peeter Joot. Fourier series vacuum maxwell's equations. "`http://sites.google.com/site/peeterjoot/math2009/fourier_series_maxwell.pdf`", a.

[Joot(b)] Peeter Joot. Electrostatic field energy. "`http://sites.google.com/site/peeterjoot/math/electric_field_energy.pdf`", b.

[Joot(c)] Peeter Joot. Energy momentum tensor. "`http://sites.google.com/site/peeterjoot/math/energy_momentum_tensor.pdf`", c.

[Joot(d)] Peeter Joot. Plane wave fourier series solutions to the maxwell vacuum equation. "`http://sites.google.com/site/peeterjoot/math2009/planewave.pdf`", d.

[Joot(e)] Peeter Joot. Poynting vector and electrodynamic energy conservation. "`http://sites.google.com/site/peeterjoot/math/poynting.pdf`", e.

[Joot(f)] Peeter Joot. Time rate of change of the poynting vector, and its conservation law. "`http://sites.google.com/site/peeterjoot/math2009/poynting_rate.pdf`", f.