Composition of rotations exersize. Two nineties.

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1 Problem.

Rotate 90 about the z-axis, and then 90 about the new x-axis (problem from Alan M's book draft).

2 Solution.

The z-axis rotation is

$$R_{z,90}(\mathbf{x}) = e^{-\mathbf{e}_{12}} \mathbf{x} e^{\mathbf{e}_{12}}$$

and the rotation about the new x-axis (ie: in the old -1,3 plane) is

$$R_{x',90}(\mathbf{x}') = e^{\mathbf{e}_{13}} \mathbf{x}' e^{-\mathbf{e}_{13}}$$

Therefore the composite rotation is

$$R(\mathbf{x}) = e^{\mathbf{e}_{13}} e^{-\mathbf{e}_{12}} \mathbf{x} e^{\mathbf{e}_{12}} e^{-\mathbf{e}_{13}}$$

We want to expand the product

$$\begin{aligned}
R &= e^{\mathbf{e}_{13}} e^{-\mathbf{e}_{12}} \\
&= \frac{1}{2}(1 + \mathbf{e}_{13})(1 - \mathbf{e}_{12}) \\
&= \frac{1}{2}(\mathbf{e}_1 - \mathbf{e}_3)\mathbf{e}_1\mathbf{e}_1(\mathbf{e}_1 - \mathbf{e}_2) \\
&= \frac{1}{2}(\mathbf{e}_1 - \mathbf{e}_3) \cdot (\mathbf{e}_1 - \mathbf{e}_2) + \frac{1}{2}(\mathbf{e}_1 - \mathbf{e}_3) \wedge (\mathbf{e}_1 - \mathbf{e}_2) \\
&= \frac{1}{2} + \frac{\sqrt{3}}{2} \frac{(\mathbf{e}_1 - \mathbf{e}_3) \wedge (\mathbf{e}_1 - \mathbf{e}_2)}{\sqrt{3}}
\end{aligned}$$

Letting $i = ((\mathbf{e}_1 - \mathbf{e}_2) \wedge (\mathbf{e}_1 - \mathbf{e}_3))/\sqrt{3}$ we have

$$
\begin{aligned}
R &= \cos(\pi/3) - i \sin(\pi/3) \\
&= e^{-i\pi/3}
\end{aligned}
$$

So, the composite rotation will take vectors that lie in the $(\mathbf{e}_1 - \mathbf{e}_2) \wedge (\mathbf{e}_1 - \mathbf{e}_3)$ plane, and rotate them by $2\pi/3 = 120^\circ$.

In terms of a normal we can write the plane in its dual form

$$
i = \tilde{I}(Ii) = -I\mathbf{n}
$$

So the normal of the rotational plane is

$$
\begin{aligned}
\mathbf{n} &= \frac{1}{\sqrt{3}} \mathbf{e}_{123} \left(-\mathbf{e}_{13} - \mathbf{e}_{21} + \mathbf{e}_{23}\right) \\
&= \frac{-1}{\sqrt{3}} \left(\mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_1\right)
\end{aligned}
$$

So we can also write this rotation as a rotation about the $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ axis (with a sense that I'd have to think about to get right).