Simple minded variation of one dimensional wave equation Lagrangian.

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From the action

$$S_\eta = \int dxdt \frac{1}{2} \left(\left(\frac{\partial \eta}{\partial x} \right)^2 - \frac{1}{v^2} \left(\frac{\partial \eta}{\partial t} \right)^2 \right)$$

We can vary the field $\eta = \psi + \epsilon$, where ψ is the field variable to be determined, and ϵ is the field variable allowed to vary within the volume of integration.

Forming the difference, subtracts off the "constant" parts of the action due only to the optimal field variable ψ

$$\begin{aligned} S_{\psi+\epsilon} - S_\psi &= \int dxdt \frac{1}{2} \left(\left(\frac{\partial(\psi+\epsilon)}{\partial x} \right)^2 - \frac{1}{v^2} \left(\frac{\partial(\psi+\epsilon)}{\partial t} \right)^2 \right) - \int dxdt \frac{1}{2} \left(\left(\frac{\partial \psi}{\partial x} \right)^2 - \frac{1}{v^2} \left(\frac{\partial \psi}{\partial t} \right)^2 \right) \\ &= \int dxdt \left(\frac{\partial \psi}{\partial x} \frac{\partial \epsilon}{\partial x} - \frac{\partial \psi}{\partial t} \frac{\partial \epsilon}{\partial t} \right) + \int dxdt \frac{1}{2} \left(\left(\frac{\partial \epsilon}{\partial x} \right)^2 - \frac{1}{v^2} \left(\frac{\partial \epsilon}{\partial t} \right)^2 \right) \end{aligned}$$

Now integrating by parts

$$S_{\psi+\epsilon} - S_\psi = \int dxdt \left(\frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial x^2} \right) \epsilon + \int dxdt \frac{1}{2} \left(-\frac{\partial^2 \epsilon}{\partial x^2} + \frac{1}{v^2} \frac{\partial^2 \epsilon}{\partial t^2} \right) \epsilon$$

Roughly speaking the terms that are quadradic in ϵ can be discarded as small, and if the remaining differential is to be zero for all ϵ, we are left with the wave equation

$$\frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial x^2} = 0$$

with solutions of the form

$$\psi = f(x \pm vt)$$