Infinite square well wavefunction.

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1. Motivation.

Work problem 4.1 from [1], calculation of the eigensolution for an infinite square well, with boundaries $[-a/2, a/2]$. It's actually a bit tidier seeming to generalize this slightly to boundaries $[a, b]$, which also implicitly solves the problem. This is surely a problem that is done in 700 other QM texts, but I liked the way I did it this time so am writing it down.

2. Guts

Our equation to solve is $i\hbar\Psi_t = -(\hbar^2/2m)\Psi_{xx}$. Separation of variables $\Psi = T\phi$ gives us

$$T \propto e^{-iEt/\hbar} \tag{1}$$

$$\phi'' = -\frac{2mE}{\hbar^2}\phi \tag{2}$$

With $k^2 = 2mE/\hbar^2$, we have

$$\phi = Ae^{ikx} + Be^{-ikx}, \tag{3}$$

and the usual $\phi(a) = \phi(b) = 0$ boundary conditions give us

$$0 = \begin{bmatrix} e^{ika} & e^{-ika} \\ e^{ikb} & e^{-ikb} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}. \tag{4}$$

We must have a zero determinant, which gives us the constraints on k immediately

$$\begin{aligned} 0 &= e^{ik(a-b)} - e^{ik(b-a)} \\ &= 2i\sin(k(a-b)). \end{aligned}$$

So our constraint on k in terms of integers n, and the corresponding integration constant E

$$k = \frac{n\pi}{b-a} \tag{5}$$

$$E = \frac{\hbar^2 n^2 \pi^2}{2m(b-a)^2}. \tag{6}$$

One of the constants A, B can be eliminated directly by picking any one of the two zeros from 4

$$\begin{aligned} &Ae^{ika} + Be^{-ika} = 0 \\ &\implies \\ &B = -Ae^{2ika} \end{aligned}$$

So we have

$$\phi = A\left(e^{ikx} - e^{ik(2a-x)}\right). \tag{7}$$

Or,

$$\phi = 2Aie^{ika}\sin(k(x-a)) \tag{8}$$

Because probability densities, currents and the expectations of any operators will always have paired ϕ and ϕ^* factors, any constant phase factors like ie^{ika} above can be dropped, or absorbed into the constant A, and we can write

$$\phi = 2A\sin(k(x-a)) \tag{9}$$

The only thing left is to fix A by integrating $|\phi|^2$, for which we have

$$\begin{aligned} 1 &= \int_a^b \phi\phi^* dx \\ &= A^2 \int_a^b dx \left(e^{ikx} - e^{ik(2a-x)}\right)\left(e^{-ikx} - e^{-ik(2a-x)}\right) \\ &= A^2 \int_a^b dx \left(2 - e^{ik(2a-2x)} - e^{ik(-2a+2x)}\right) \\ &= 2A^2 \int_a^b dx \left(1 - \cos(2k(a-x))\right) \end{aligned}$$

This last trig term vanishes over the integration region and we are left with

$$A = \frac{1}{\sqrt{2(b-a)}}, \tag{10}$$

which essentially completes the problem. A final substitution back into 8 allows for a final tidy up

$$\phi = \sqrt{\frac{2}{b-a}}\sin(k(x-a)). \tag{11}$$

References

[1] R. Liboff. *Introductory quantum mechanics*. Cambridge: Addison-Wesley Press, Inc, 2003. 1