A problem on spherical harmonics.

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Peeter Joot — peeter.joot@gmail.com
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Contents

1. Motivation.

One of the PHY356 exam questions from the final I recall screwing up on, and figuring it out after the fact on the drive home. The question actually clarified a difficulty I'd had, but unfortunately I hadn't had the good luck to perform such a question, to help figure this out before the exam.

From what I recall the question provided an initial state, with some degeneracy in m, perhaps of the following form

$$|\phi(0)\rangle = \sqrt{\frac{1}{7}}|12\rangle + \sqrt{\frac{2}{7}}|10\rangle + \sqrt{\frac{4}{7}}|20\rangle, \tag{1}$$

and a Hamiltonian of the form

$$H = \alpha L_z \tag{2}$$

From what I recall of the problem, I am going to reattempt it here now.

1.1. *Evolved state.*

One part of the question was to calculate the evolved state. Application of the time evolution operator gives us

$$|\phi(t)\rangle = e^{-i\alpha L_z t/\hbar} \left(\sqrt{\frac{1}{7}}|12\rangle + \sqrt{\frac{2}{7}}|10\rangle + \sqrt{\frac{4}{7}}|20\rangle \right). \tag{3}$$

Now we note that $L_z|12\rangle = 2\hbar|12\rangle$, and $L_z|l0\rangle = 0|l0\rangle$, so the exponentials reduce this nicely to just

$$|\phi(t)\rangle = \sqrt{\frac{1}{7}}e^{-2i\alpha t}|12\rangle + \sqrt{\frac{2}{7}}|10\rangle + \sqrt{\frac{4}{7}}|20\rangle. \tag{4}$$

1.2. *Probabilities for L_z measurement outcomes.*

I believe we were also asked what the probabilities for the outcomes of a measurement of L_z at this time would be. Here is one place that I think that I messed up, and it is really a translation error, attempting to get from the english description of the problem to the math description of the same. I'd had trouble with this process a few times in the problems, and managed to blunder through use of language like "measure", and "outcome", but don't think I really understood how these were used properly.

What are the outcomes that we measure? We measure operators, but the result of a measurement is the eigenvalue associated with the operator. What are the eigenvalues of the L_z operator? These are the $m\hbar$ values, from the operation $L_z|lm\rangle = m\hbar|lm\rangle$. So, given this initial state, there are really two outcomes that are possible, since we have two distinct eigenvalues. These are $2\hbar$ and 0 for $m = 2$, and $m = 0$ respectively.

A measurement of the "outcome" $2\hbar$, will be the probability associated with the amplitude $\langle 12|\phi(t)\rangle$ (ie: the absolute square of this value). That is

$$|\langle 12|\phi(t)\rangle|^2 = \frac{1}{7}. \tag{5}$$

Now, the only other outcome for a measurement of L_z for this state is a measurement of $0\hbar$, and the probability of this is then just $1 - \frac{1}{7} = \frac{6}{7}$. On the exam, I think I listed probabilities for three outcomes, with values $\frac{1}{7}, \frac{2}{7}, \frac{4}{7}$ respectively, but in retrospect that seems blatently wrong.

1.3. *Probabilities for $\mathbf{L}^2$ measurement outcomes.*

What are the probabilities for the outcomes for a measurement of $\mathbf{L}^2$ after this? The first question is really what are the outcomes. That's really a question of what are the possible eigenvalues of $\mathbf{L}^2$ that can be measured at this point. Recall that we have

$$\mathbf{L}^2|lm\rangle = \hbar^2 l(l+1)|lm\rangle \tag{6}$$

So for a state that has only $l = 1, 2$ contributions before the measurement, the eigenvalues that can be observed for the $\mathbf{L}^2$ operator are respectively $2\hbar^2$ and $6\hbar^2$ respectively.

For the $l = 2$ case, our probability is $4/7$, leaving $3/7$ as the probability for measurement of the $l = 1$ ($2\hbar^2$) eigenvalue. We can compute this two ways, and it seems worthwhile to consider both. This first method makes use of the fact that the L_z operator leaves the state vector intact, but it also seems like a bit of a cheat. Consider instead two possible results of measurement after the L_z observation. When an L_z measurement of $0\hbar$ is performed our state will be left with only the $m = 0$ kets. That is

$$|\psi_a\rangle = \frac{1}{\sqrt{3}}\left(|10\rangle + \sqrt{2}|20\rangle\right), \tag{7}$$

whereas, when a $2\hbar$ measurement of L_z is performed our state would then only have the $m = 2$ contribution, and would be

$$|\psi_b\rangle = e^{-2i\alpha t}|12\rangle. \tag{8}$$

We have two possible ways of measuring the $2\hbar^2$ eigenvalue for $\mathbf{L}^2$. One is when our state was $|\psi_a\rangle$ (, and the resulting state has a $|10\rangle$ component, and the other is after the $m = 2$ measurement, where our state is left with a $|12\rangle$ component.

The resulting probability is then a conditional probability result

$$\frac{6}{7}|\langle 10|\psi_a\rangle|^2 + \frac{1}{7}|\langle 12|\psi_b\rangle|^2 = \frac{3}{7} \tag{9}$$

The result is the same, as expected, but this is likely a more convicing argument.