Verifying the Helmholtz Green's function.

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Contents

1. Motivation.

In class this week, looking at an instance of the Helmholtz equation

$$\left(\boldsymbol{\nabla}^2 + \mathbf{k}^2\right) \psi_{\mathbf{k}}(\mathbf{r}) = s(\mathbf{r}). \tag{1}$$

We were told that the Green's function

$$\left(\boldsymbol{\nabla}^2 + \mathbf{k}^2\right) G^0(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') \tag{2}$$

that can be used to solve for a particular solution this differential equation via convolution

$$\psi_{\mathbf{k}}(\mathbf{r}) = \int G^0(\mathbf{r}, \mathbf{r}') s(\mathbf{r}') d^3\mathbf{r}', \tag{3}$$

had the value

$$G^0(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi} \frac{e^{ik\lvert\mathbf{r}-\mathbf{r}'\rvert}}{\lvert\mathbf{r} - \mathbf{r}'\rvert}. \tag{4}$$

Let's try to verify this.

2. Guts

Application of the Helmholtz differential operator $\boldsymbol{\nabla}^2 + \mathbf{k}^2$ on the presumed solution gives

$$(\boldsymbol{\nabla}^2 + \mathbf{k}^2)\psi_{\mathbf{k}}(\mathbf{r}) = -\frac{1}{4\pi} \int (\boldsymbol{\nabla}^2 + \mathbf{k}^2) \frac{e^{ik\lvert\mathbf{r}-\mathbf{r}'\rvert}}{\lvert\mathbf{r} - \mathbf{r}'\rvert} s(\mathbf{r}') d^3\mathbf{r}'. \tag{5}$$

2.1. *When* $\mathbf{r} \neq \mathbf{r}'$.

To proceed we'll need to evaluate

$$\boldsymbol{\nabla}^2 \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}. \tag{6}$$

Writing $\mu = |\mathbf{r}-\mathbf{r}'|$ we start with the computation of

$$\begin{aligned}
\frac{\partial}{\partial x}\frac{e^{ik\mu}}{\mu} &= \frac{\partial \mu}{\partial x}\left(\frac{ik}{\mu} - \frac{1}{\mu^2}\right)e^{ik\mu} \\
&= \frac{\partial \mu}{\partial x}\left(ik - \frac{1}{\mu}\right)\frac{e^{ik\mu}}{\mu}
\end{aligned}$$

We see that we'll have

$$\boldsymbol{\nabla}\frac{e^{ik\mu}}{\mu} = \left(ik - \frac{1}{\mu}\right)\frac{e^{ik\mu}}{\mu}\boldsymbol{\nabla}\mu. \tag{7}$$

Taking second derivatives with respect to x we find

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}\frac{e^{ik\mu}}{\mu} &= \frac{\partial^2 \mu}{\partial x^2}\left(ik - \frac{1}{\mu}\right)\frac{e^{ik\mu}}{\mu} + \frac{\partial \mu}{\partial x}\frac{\partial \mu}{\partial x}\frac{1}{\mu^2}\frac{e^{ik\mu}}{\mu} + \left(\frac{\partial \mu}{\partial x}\right)^2\left(ik - \frac{1}{\mu}\right)^2\frac{e^{ik\mu}}{\mu} \\
&= \frac{\partial^2 \mu}{\partial x^2}\left(ik - \frac{1}{\mu}\right)\frac{e^{ik\mu}}{\mu} + \left(\frac{\partial \mu}{\partial x}\right)^2\left(-k^2 - \frac{2ik}{\mu} + \frac{2}{\mu^2}\right)\frac{e^{ik\mu}}{\mu}.
\end{aligned}$$

Our Laplacian is then

$$\boldsymbol{\nabla}^2\frac{e^{ik\mu}}{\mu} = \left(ik - \frac{1}{\mu}\right)\frac{e^{ik\mu}}{\mu}\boldsymbol{\nabla}^2\mu + \left(-k^2 - \frac{2ik}{\mu} + \frac{2}{\mu^2}\right)\frac{e^{ik\mu}}{\mu}(\boldsymbol{\nabla}\mu)^2. \tag{8}$$

Now lets calculate the derivatives of μ. Working on x again, we have

$$\begin{aligned}
\frac{\partial}{\partial x}\mu &= \frac{\partial}{\partial x}\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \\
&= \frac{1}{2}2(x-x')\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \\
&= \frac{x-x'}{\mu}.
\end{aligned}$$

So we have

$$\boldsymbol{\nabla}\mu = \frac{\mathbf{r}-\mathbf{r}'}{\mu} \tag{9}$$

$$(\boldsymbol{\nabla}\mu)^2 = 1 \tag{10}$$

Taking second derivatives with respect to x we find

$$
\begin{aligned}
\frac{\partial^2}{\partial x^2}\mu &= \frac{\partial}{\partial x}\frac{x-x'}{\mu} \\
&= \frac{1}{\mu} - (x-x')\frac{\partial \mu}{\partial x}\frac{1}{\mu^2} \\
&= \frac{1}{\mu} - (x-x')\frac{x-x'}{\mu}\frac{1}{\mu^2} \\
&= \frac{1}{\mu} - (x-x')^2\frac{1}{\mu^3}.
\end{aligned}
$$

So we find

$$\boldsymbol{\nabla}^2\mu = \frac{3}{\mu} - \frac{1}{\mu}, \tag{11}$$

or

$$\boldsymbol{\nabla}^2\mu = \frac{2}{\mu}. \tag{12}$$

Inserting this and $(\boldsymbol{\nabla}\mu)^2$ into 8 we find

$$\boldsymbol{\nabla}^2\frac{e^{ik\mu}}{\mu} = \left(ik - \frac{1}{\mu}\right)\frac{e^{ik\mu}}{\mu}\frac{2}{\mu} + \left(-k^2 - \frac{2ik}{\mu} + \frac{2}{\mu^2}\right)\frac{e^{ik\mu}}{\mu} \quad = -k^2\frac{e^{ik\mu}}{\mu} \tag{13}$$

This shows us that provided $\mathbf{r} \neq \mathbf{r}'$ we have

$$(\boldsymbol{\nabla}^2 + \mathbf{k}^2)G^0(\mathbf{r}, \mathbf{r}') = 0. \tag{14}$$

2.2. *In the neighborhood of* $|\mathbf{r} - \mathbf{r}'| < \epsilon$.

Having shown that we end up with zero everywhere that $\mathbf{r} \neq \mathbf{r}'$ we are left to consider a neighborhood of the volume surrounding the point $\mathbf{r}$ in our integral. Following the Coulomb treatment in §2.2 of [1] we use a spherical volume element centered around $\mathbf{r}$ of radius ϵ, and then convert a divergence to a surface area to evaluate the integral away from the problematic point

$$-\frac{1}{4\pi}\int_{\text{all space}}(\boldsymbol{\nabla}^2 + \mathbf{k}^2)\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}s(\mathbf{r}')d^3\mathbf{r}' = -\frac{1}{4\pi}\int_{|\mathbf{r}-\mathbf{r}'|<\epsilon}(\boldsymbol{\nabla}^2 + \mathbf{k}^2)\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}s(\mathbf{r}')d^3\mathbf{r}' \tag{15}$$

We make the change of variables $\mathbf{r}' = \mathbf{r} + \mathbf{a}$. We add an explicit $\mathbf{r}$ suffix to our Laplacian at the same time to remind us that it is taking derivatives with respect to the coordinates of $\mathbf{r} = (x, y, z)$, and not the coordinates of our integration variable $\mathbf{a} = (a_x, a_y, a_z)$. Assuming sufficient continuity and "well behavedness" of $s(\mathbf{r}')$ we'll be able to pull it out of the integral, giving

$$
\begin{aligned}
-\frac{1}{4\pi}\int_{|\mathbf{r}-\mathbf{r}'|<\epsilon}(\boldsymbol{\nabla}_{\mathbf{r}}^2 + \mathbf{k}^2)\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}s(\mathbf{r}')d^3\mathbf{r}' &= -\frac{1}{4\pi}\int_{|\mathbf{a}|<\epsilon}(\boldsymbol{\nabla}_{\mathbf{r}}^2 + \mathbf{k}^2)\frac{e^{ik|\mathbf{a}|}}{|\mathbf{a}|}s(\mathbf{r}+\mathbf{a})d^3\mathbf{a} \\
&= -\frac{s(\mathbf{r})}{4\pi}\int_{|\mathbf{a}|<\epsilon}(\boldsymbol{\nabla}_{\mathbf{r}}^2 + \mathbf{k}^2)\frac{e^{ik|\mathbf{a}|}}{|\mathbf{a}|}d^3\mathbf{a}
\end{aligned}
$$

Recalling the dependencies on the derivatives of $|\mathbf{r}-\mathbf{r}'|$ in our previous gradient evaluations, we note that we have

$$\nabla_{\mathbf{r}}\left|\mathbf{r}-\mathbf{r}'\right| = -\nabla_{\mathbf{a}}|\mathbf{a}| \tag{16}$$

$$(\nabla_{\mathbf{r}}\left|\mathbf{r}-\mathbf{r}'\right|)^2 = (\nabla_{\mathbf{a}}|\mathbf{a}|)^2 \tag{17}$$

$$\nabla_{\mathbf{r}}^2\left|\mathbf{r}-\mathbf{r}'\right| = \nabla_{\mathbf{a}}^2|\mathbf{a}|, \tag{18}$$

so with $\mathbf{a} = \mathbf{r}-\mathbf{r}'$, we can rewrite our Laplacian as

$$\nabla_{\mathbf{r}}^2 \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} = \nabla_{\mathbf{a}}^2 \frac{e^{ik|\mathbf{a}|}}{|\mathbf{a}|} = \nabla_{\mathbf{a}} \cdot \left(\nabla_{\mathbf{a}} \frac{e^{ik|\mathbf{a}|}}{|\mathbf{a}|} \right) \tag{19}$$

This gives us

$$\begin{aligned}
-\frac{s(\mathbf{r})}{4\pi}\int_{|\mathbf{a}|<\epsilon} (\nabla_{\mathbf{a}}^2 + \mathbf{k}^2)\frac{e^{ik|\mathbf{a}|}}{|\mathbf{a}|} d^3\mathbf{a} &= -\frac{s(\mathbf{r})}{4\pi}\int_{dV} \nabla_{\mathbf{a}} \cdot \left(\nabla_{\mathbf{a}} \frac{e^{ik|\mathbf{a}|}}{|\mathbf{a}|} \right) d^3\mathbf{a} - \frac{s(\mathbf{r})}{4\pi}\int_{dV} \mathbf{k}^2 \frac{e^{ik|\mathbf{a}|}}{|\mathbf{a}|} d^3\mathbf{a} \\
&= -\frac{s(\mathbf{r})}{4\pi}\int_{dA} \left(\nabla_{\mathbf{a}} \frac{e^{ik|\mathbf{a}|}}{|\mathbf{a}|} \right) \cdot \hat{\mathbf{a}} d^2\mathbf{a} - \frac{s(\mathbf{r})}{4\pi}\int_{dV} \mathbf{k}^2 \frac{e^{ik|\mathbf{a}|}}{|\mathbf{a}|} d^3\mathbf{a}
\end{aligned}$$

To complete these evaluations, we can now employ a spherical coordinate change of variables. Let's do the $\mathbf{k}^2$ volume integral first. We have

$$\begin{aligned}
\int_{dV} \mathbf{k}^2 \frac{e^{ik|\mathbf{a}|}}{|\mathbf{a}|} d^3\mathbf{a} &= \int_{a=0}^{\epsilon}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \mathbf{k}^2 \frac{e^{ika}}{a} a^2 da \sin\theta d\theta d\phi \\
&= 4\pi k^2 \int_{a=0}^{\epsilon} a e^{ika} da \\
&= 4\pi \int_{u=0}^{k\epsilon} u e^{iu} du \\
&= 4\pi (-iu+1) e^{iu}\Big|_0^{k\epsilon} \\
&= 4\pi \left((-ik\epsilon + 1) e^{ik\epsilon} - 1 \right)
\end{aligned}$$

To evaluate the surface integral we note that we'll require only the radial portion of the gradient, so have

$$\begin{aligned}
\left(\nabla_{\mathbf{a}} \frac{e^{ik|\mathbf{a}|}}{|\mathbf{a}|} \right) \cdot \hat{\mathbf{a}} &= \left(\hat{\mathbf{a}} \frac{\partial}{\partial a} \frac{e^{ika}}{a} \right) \cdot \hat{\mathbf{a}} \\
&= \frac{\partial}{\partial a} \frac{e^{ika}}{a} \\
&= \left(ik\frac{1}{a} - \frac{1}{a^2} \right) e^{ika} \\
&= (ika - 1) \frac{e^{ika}}{a^2}
\end{aligned}$$

Our area element is $a^2 \sin\theta d\theta d\phi$, so we are left with

$$\begin{aligned}\int_{dA} \left(\boldsymbol{\nabla}_{\mathbf{a}} \frac{e^{ik|\mathbf{a}|}}{|\mathbf{a}|} \right) \cdot \hat{\mathbf{a}} d^2\mathbf{a} &= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} (ika - 1) \frac{e^{ika}}{a^2} a^2 \sin\theta d\theta d\phi \bigg|_{a=\epsilon} \\ &= 4\pi \left(ik\epsilon - 1\right) e^{ik\epsilon}\end{aligned} \tag{20}$$

Putting everything back together we have

$$\begin{aligned}-\frac{1}{4\pi} \int_{\text{all space}} (\boldsymbol{\nabla}^2 + \mathbf{k}^2) \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} s(\mathbf{r}') d^3\mathbf{r}' &= -s(\mathbf{r}) \left((-ik\epsilon + 1) e^{ik\epsilon} - 1 + \left(ik\epsilon - 1\right) e^{ik\epsilon} \right) \\ &= -s(\mathbf{r}) \left((-ik\epsilon + 1 + ik\epsilon - 1) e^{ik\epsilon} - 1 \right)\end{aligned}$$

But this is just

$$-\frac{1}{4\pi} \int_{\text{all space}} (\boldsymbol{\nabla}^2 + \mathbf{k}^2) \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} s(\mathbf{r}') d^3\mathbf{r}' = s(\mathbf{r}). \tag{21}$$

This completes the desired verification of the Green's function for the Helmholtz operator. Observe the perfect cancellation here, so the limit of $\epsilon \rightarrow 0$ can be independent of how large k is made. You have to complete the integrals for both the Laplacian and the $\mathbf{k}^2$ portions of the integrals and add them, before taking any limits, or else you'll get into trouble (as I did in my first attempt).

References

[1] M. Schwartz. *Principles of Electrodynamics*. Dover Publications, 1987. 2.2