Some exam reflection.

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Contents

1. Charged particle in a circle.

From the 2008 PHY353 exam, given a particle of charge q moving in a circle of radius a at constant angular frequency ω.

- Find the Lienard-Wiechert potentials for points on the z-axis.
- Find the electric and magnetic fields at the center.

When I tried this I did it for points not just on the z-axis. It turns out that we also got this question on the exam (but stated slightly differently). Since I'll not get to see my exam solution again, let's work through this at a leisurely rate, and see if things look right. The problem as stated in this old practice exam is easier since it doesn't say to calculate the fields from the four potentials, so there was nothing preventing one from just grinding away and plugging stuff into the Lienard-Wiechert equations for the fields (as I did when I tried it for practice).

1.1. *The potentials.*

Let's set up our coordinate system in cylindrical coordinates. For the charged particle and the point that we measure the field, with $i = \mathbf{e}_1 \mathbf{e}_2$

$$\mathbf{x}(t) = a \mathbf{e}_1 e^{i\omega t} \tag{1}$$
$$\mathbf{r} = z \mathbf{e}_3 + \rho \mathbf{e}_1 e^{i\phi} \tag{2}$$

Here I'm using the geometric product of vectors (if that's unfamiliar then just substitute

$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \rightarrow \{\sigma_1, \sigma_2, \sigma_3\} \tag{3}$$

We can do that since the Pauli matrices also have the same semantics (with a small difference since the geometric square of a unit vector is defined as the unit scalar, whereas the Pauli matrix square is the identity matrix). The semantics we require of this vector product are just $\mathbf{e}_\alpha^2 = 1$ and $\mathbf{e}_\alpha \mathbf{e}_\beta = -\mathbf{e}_\beta \mathbf{e}_\alpha$ for any $\alpha \neq \beta$.

I'll also be loose with notation and use $\mathrm{Re}(X) = \langle X \rangle$ to select the scalar part of a multivector (or with the Pauli matrices, the portion proportional to the identity matrix).

Our task is to compute the Lienard-Wiechert potentials. Those are

$$A^0 = \frac{q}{R^*} \tag{4}$$

$$\mathbf{A} = A^0 \frac{\mathbf{v}}{c}, \tag{5}$$

where

$$\mathbf{R} = \mathbf{r} - \mathbf{x}(t_r) \tag{6}$$

$$R = |\mathbf{R}| = c(t - t_r) \tag{7}$$

$$R^* = R - \frac{\mathbf{v}}{c} \cdot \mathbf{R} \tag{8}$$

$$\mathbf{v} = \frac{d\mathbf{x}}{dt_r}. \tag{9}$$

We'll need (eventually)

$$\mathbf{v} = a\omega \mathbf{e}_2 e^{i\omega t_r} = a\omega(-\sin \omega t_r, \cos \omega t_r, 0) \tag{10}$$

$$\dot{\mathbf{v}} = -a\omega^2 \mathbf{e}_1 e^{i\omega t_r} = -a\omega^2(\cos \omega t_r, \sin \omega t_r, 0) \tag{11}$$

and also need our retarded distance vector

$$\mathbf{R} = z\mathbf{e}_3 + \mathbf{e}_1(\rho e^{i\phi} - a e^{i\omega t_r}), \tag{12}$$

From this we have

$$\begin{aligned}
R^2 &= z^2 + \left| \mathbf{e}_1(\rho e^{i\phi} - a e^{i\omega t_r}) \right|^2 \\
&= z^2 + \rho^2 + a^2 - 2\rho a (\mathbf{e}_1 \rho e^{i\phi}) \cdot (\mathbf{e}_1 e^{i\omega t_r}) \\
&= z^2 + \rho^2 + a^2 - 2\rho a \,\mathrm{Re}(e^{i(\phi - \omega t_r)}) \\
&= z^2 + \rho^2 + a^2 - 2\rho a \cos(\phi - \omega t_r)
\end{aligned}$$

So

$$R = \sqrt{z^2 + \rho^2 + a^2 - 2\rho a \cos(\phi - \omega t_r)}. \tag{13}$$

Next we need

$$
\begin{aligned}
\mathbf{R} \cdot \mathbf{v}/c &= (z\mathbf{e}_3 + \mathbf{e}_1(\rho e^{i\phi} - a e^{i\omega t_r})) \cdot \left(a \frac{\omega}{c} \mathbf{e}_2 e^{i\omega t_r} \right) \\
&= a \frac{\omega}{c} \operatorname{Re}(i(\rho e^{-i\phi} - a e^{-i\omega t_r}) e^{i\omega t_r}) \\
&= a \frac{\omega}{c} \rho \operatorname{Re}(i e^{-i\phi + i\omega t_r}) \\
&= a \frac{\omega}{c} \rho \sin(\phi - \omega t_r)
\end{aligned}
$$

So we have

$$
R^* = \sqrt{z^2 + \rho^2 + a^2 - 2\rho a \cos(\phi - \omega t_r)} - a \frac{\omega}{c} \rho \sin(\phi - \omega t_r) \tag{14}
$$

Writing $k = \omega/c$, and having a peek back at 4, our potentials are now solved for

$$
\boxed{
\begin{aligned}
A^0 &= \frac{q}{\sqrt{z^2 + \rho^2 + a^2 - 2\rho a \cos(\phi - kct_r)}} \\
\mathbf{A} &= A^0 ak(-\sin kct_r, \cos kct_r, 0).
\end{aligned}
} \tag{15}
$$

The caveat is that t_r is only specified implicitly, according to

$$
\boxed{ct_r = ct - \sqrt{z^2 + \rho^2 + a^2 - 2\rho a \cos(\phi - kct_r)}.} \tag{16}
$$

There doesn't appear to be much hope of solving for t_r explicitly in closed form.

1.2. *General fields for this system.*

With

$$
\mathbf{R}^* = \mathbf{R} - \frac{\mathbf{v}}{c} R, \tag{17}
$$

the fields are

$$
\boxed{
\begin{aligned}
\mathbf{E} &= q(1 - \mathbf{v}^2/c^2) \frac{\mathbf{R}^*}{R^{*3}} + \frac{q}{R^{*3}} \mathbf{R} \times (\mathbf{R}^* \times \dot{\mathbf{v}}/c^2) \\
\mathbf{B} &= \frac{\mathbf{R}}{R} \times \mathbf{E}.
\end{aligned}
} \tag{18}
$$

In there we have

$$
1 - \mathbf{v}^2/c^2 = 1 - a^2 \frac{\omega^2}{c^2} = 1 - a^2 k^2 \tag{19}
$$

and

$$
\begin{aligned}
\mathbf{R}^* &= z\mathbf{e}_3 + \mathbf{e}_1(\rho e^{i\phi} - a e^{ikct_r}) - ak\mathbf{e}_2 e^{ikct_r} R \\
&= z\mathbf{e}_3 + \mathbf{e}_1(\rho e^{i\phi} - a(1 - kRi) e^{ikct_r})
\end{aligned}
$$

Writing this out in coordinates isn't particularly illuminating, but can be done for completeness without too much trouble

$$\mathbf{R}^* = (\rho \cos \phi - a \cos t_r + akR \sin t_r, \rho \sin \phi - a \sin t_r - akR \cos t_r, z) \tag{20}$$

In one sense the problem could be considered solved, since we have all the pieces of the puzzle. The outstanding question is whether or not the resulting mess can be simplified at all. Let's see if the cross product reduces at all. Using

$$\mathbf{R} \times (\mathbf{R}^* \times \dot{\mathbf{v}}/c^2) = \mathbf{R}^*(\mathbf{R} \cdot \dot{\mathbf{v}}/c^2) - \frac{\dot{\mathbf{v}}}{c^2}(\mathbf{R} \cdot \mathbf{R}^*) \tag{21}$$

Perhaps one or more of these dot products can be simplified? One of them does reduce nicely

$$\begin{aligned}
\mathbf{R}^* \cdot \mathbf{R} &= (\mathbf{R} - R\mathbf{v}/c) \cdot \mathbf{R} \\
&= R^2 - (\mathbf{R} \cdot \mathbf{v}/c)R \\
&= R^2 - Rak\rho \sin(\phi - kct_r) \\
&= R(R - ak\rho \sin(\phi - kct_r))
\end{aligned}$$

$$\begin{aligned}
\mathbf{R} \cdot \dot{\mathbf{v}}/c^2 &= \Big(z\mathbf{e}_3 + \mathbf{e}_1(\rho e^{i\phi} - ae^{i\omega t_r})\Big) \cdot (-ak^2 \mathbf{e}_1 e^{i\omega t_r}) \\
&= -ak^2 \Big\langle \mathbf{e}_1(\rho e^{i\phi} - ae^{i\omega t_r})\mathbf{e}_1 e^{i\omega t_r}) \Big\rangle \\
&= -ak^2 \Big\langle (\rho e^{i\phi} - ae^{i\omega t_r})e^{-i\omega t_r}) \Big\rangle \\
&= -ak^2 \Big\langle \rho e^{i\phi - i\omega t_r} - a \Big\rangle \\
&= -ak^2(\rho \cos(\phi - kct_r) - a)
\end{aligned}$$

Putting this cross product back together we have

$$\begin{aligned}
\mathbf{R} \times (\mathbf{R}^* \times \dot{\mathbf{v}}/c^2) &= ak^2(a - \rho \cos(\phi - kct_r))\mathbf{R}^* + ak^2 \mathbf{e}_1 e^{ikct_r} R(R - ak\rho \sin(\phi - kct_r)) \\
&= ak^2(a - \rho \cos(\phi - kct_r)) \Big(z\mathbf{e}_3 + \mathbf{e}_1(\rho e^{i\phi} - a(1 - kRi)e^{ikct_r})\Big) \\
&\quad + ak^2 R \mathbf{e}_1 e^{ikct_r}(R - ak\rho \sin(\phi - kct_r))
\end{aligned}$$

Writing

$$\phi_r = \phi - kct_r, \tag{22}$$

this can be grouped into similar terms

$$\begin{aligned}
\mathbf{R} \times (\mathbf{R}^* \times \dot{\mathbf{v}}/c^2) &= ak^2(a - \rho \cos \phi_r)z\mathbf{e}_3 \\
&\quad + ak^2 \mathbf{e}_1(a - \rho \cos \phi_r)\rho e^{i\phi} \\
&\quad + ak^2 \mathbf{e}_1 \left(-a(a - \rho \cos \phi_r)(1 - kRi) + R(R - ak\rho \sin \phi_r)\right) e^{ikct_r}
\end{aligned} \tag{23}$$

The electric field pieces can now be collected. Not expanding out the R^* from 14, this is

$$
\begin{aligned}
\mathbf{E} &= \frac{q}{(R^*)^3} z\mathbf{e}_3 \left(1 - a\rho k^2 \cos\phi_r\right) \\
&+ \frac{q}{(R^*)^3} \rho\mathbf{e}_1 \left(1 - a\rho k^2 \cos\phi_r\right) e^{i\phi} \\
&+ \frac{q}{(R^*)^3} a\mathbf{e}_1 \left(-\left(1 + ak^2(a - \rho\cos\phi_r)\right)(1 - kRi)(1 - a^2k^2) + k^2R(R - ak\rho\sin\phi_r)\right) e^{ikct_r}
\end{aligned}
\tag{24}
$$

Along the z-axis where $\rho = 0$ what do we have?

$$R = \sqrt{z^2 + a^2} \tag{25a}$$

$$A^0 = \frac{q}{R} \tag{25b}$$

$$\mathbf{A} = A^0 ak\mathbf{e}_2 e^{ikct_r} \tag{25c}$$

$$ct_r = ct - \sqrt{z^2 + a^2} \tag{25d}$$

$$
\begin{aligned}
\mathbf{E} &= \frac{q}{R^3} z\mathbf{e}_3 \\
&+ \frac{q}{R^3} a\mathbf{e}_1 \left(-(1 - a^4k^4)(1 - kRi) + k^2R^2\right) e^{ikct_r}
\end{aligned}
\tag{25e}
$$

$$\mathbf{B} = \frac{z\mathbf{e}_3 - a\mathbf{e}_1 e^{ikct_r}}{R} \times \mathbf{E} \tag{25f}$$

The magnetic term here looks like it can be reduced a bit.

1.3. *An approximation near the center.*

Unlike the old exam I did, where it didn't specify that the potentials had to be used to calculate the fields, and the problem was reduced to one of algebraic manipulation, our exam explicitly asked for the potentials to be used to calculate the fields.

There was also the restriction to compute them near the center. Setting $\rho = 0$ so that we are looking only near the z-axis, we have

$$A^0 = \frac{q}{\sqrt{z^2 + a^2}} \tag{26}$$

$$\mathbf{A} = \frac{qak\mathbf{e}_2 e^{ikct_r}}{\sqrt{z^2 + a^2}} = \frac{qak(-\sin kct_r, \cos kct_r, 0)}{\sqrt{z^2 + a^2}} \tag{27}$$

$$t_r = t - R/c = t - \sqrt{z^2 + a^2}/c \tag{28}$$

Now we are set to calculate the electric and magnetic fields directly from these. Observe that we have a spatial dependence in due to the t_r quantities and that will have an effect when we operate with the gradient.

In the exam I'd asked Simon (our TA) if this question was asking for the fields at the origin (ie: in the plane of the charge's motion in the center) or along the z-axis. He said in the plane. That would simplify things, but perhaps too much since A^0 becomes constant (in my exam attempt I somehow fudged this to get what I wanted for the $v = 0$ case, but that must have been wrong, and was the result of rushed work).

Let's now proceed with the field calculation from these potentials

$$\mathbf{E} = -\boldsymbol{\nabla} A^0 - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \tag{29}$$

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{30}$$

For the electric field we need

$$\begin{aligned}
\boldsymbol{\nabla} A^0 &= q\mathbf{e}_3 \partial_z (z^2 + a^2)^{-1/2} \\
&= -q\mathbf{e}_3 \frac{z}{(\sqrt{z^2 + a^2})^3},
\end{aligned}$$

and

$$\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = \frac{qak^2 \mathbf{e}_2 \mathbf{e}_1 \mathbf{e}_2 e^{ikct_r}}{\sqrt{z^2 + a^2}}. \tag{31}$$

Putting these together, our electric field near the z-axis is

$$\mathbf{E} = q\mathbf{e}_3 \frac{z}{(\sqrt{z^2 + a^2})^3} + \frac{qak^2 \mathbf{e}_1 e^{ikct_r}}{\sqrt{z^2 + a^2}}. \tag{32}$$

(another mistake I made on the exam, since I somehow fooled myself into forcing what I knew had to be in the gradient term, despite having essentially a constant scalar potential (having taken $z = 0$)).

What do we get for the magnetic field. In that case we have

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{A}(z) &= \mathbf{e}_\alpha \times \partial_\alpha \mathbf{A} \\
&= \mathbf{e}_3 \times \partial_z \frac{qak\mathbf{e}_2 e^{ikct_r}}{\sqrt{z^2 + a^2}} \\
&= \mathbf{e}_3 \times (\mathbf{e}_2 e^{ikct_r}) qak \frac{\partial}{\partial z} \frac{1}{\sqrt{z^2 + a^2}} + qak \frac{1}{\sqrt{z^2 + a^2}} \mathbf{e}_3 \times (\mathbf{e}_2 \partial_z e^{ikct_r}) \\
&= -\mathbf{e}_3 \times (\mathbf{e}_2 e^{ikct_r}) qak \frac{z}{(\sqrt{z^2 + a^2})^3} + qak \frac{1}{\sqrt{z^2 + a^2}} \mathbf{e}_3 \times \left(\mathbf{e}_2 \mathbf{e}_1 \mathbf{e}_2 kce^{ikct_r} \partial_z (t - \sqrt{z^a + a^2}/c) \right) \\
&= -\mathbf{e}_3 \times (\mathbf{e}_2 e^{ikct_r}) qak \frac{z}{(\sqrt{z^2 + a^2})^3} - qak^2 \frac{z}{z^2 + a^2} \mathbf{e}_3 \times \left(\mathbf{e}_1 k e^{ikct_r} \right) \\
&= -\frac{qakz\mathbf{e}_3}{z^2 + a^2} \times \left(\frac{\mathbf{e}_2 e^{ikct_r}}{\sqrt{z^2 + a^2}} + k\mathbf{e}_1 e^{ikct_r} \right)
\end{aligned}$$

For the direction vectors in the cross products above we have

$$\begin{aligned}
\mathbf{e}_3 \times (\mathbf{e}_2 e^{i\mu}) &= \mathbf{e}_3 \times (\mathbf{e}_2 \cos\mu - \mathbf{e}_1 \sin\mu) \\
&= -\mathbf{e}_1 \cos\mu - \mathbf{e}_2 \sin\mu \\
&= -\mathbf{e}_1 e^{i\mu}
\end{aligned}$$

and

$$
\begin{aligned}
\mathbf{e}_3 \times (\mathbf{e}_1 e^{i\mu}) &= \mathbf{e}_3 \times (\mathbf{e}_1 \cos\mu + \mathbf{e}_2 \sin\mu) \\
&= \mathbf{e}_2 \cos\mu - \mathbf{e}_1 \sin\mu \\
&= \mathbf{e}_2 e^{i\mu}
\end{aligned}
$$

Putting everything, and summarizing results for the fields, we have

$$
\mathbf{E} = q\mathbf{e}_3 \frac{z}{(\sqrt{z^2+a^2})^3} + \frac{qak^2 \mathbf{e}_1 e^{i\omega t_r}}{\sqrt{z^2+a^2}} \tag{33}
$$

$$
\mathbf{B} = \frac{qakz}{z^2+a^2} \left(\frac{\mathbf{e}_1}{\sqrt{z^2+a^2}} - k\mathbf{e}_2 \right) e^{i\omega t_r} \tag{34}
$$

The electric field expression above compares well to 25e. We have the Coulomb term and the radiation term. It is harder to compare the magnetic field to the exact result 25f since I did not expand that out.

FIXME: A question to consider. If all this worked should we not also get

$$
\mathbf{B} \stackrel{?}{=} \frac{z\mathbf{e}_3 - \mathbf{e}_1 a e^{i\omega t_r}}{\sqrt{z^2+a^2}} \times \mathbf{E}. \tag{35}
$$

However, if I do this check I get

$$
\mathbf{B} = \frac{qaz}{z^2+a^2} \left(\frac{1}{z^2+a^2} + k^2 \right) \mathbf{e}_2 e^{i\omega t_r}. \tag{36}
$$

1.4. *Without geometric algebra.*

I tried the problem of calculating the Lienard-Wiechert potentials for circular motion once again in [1] but with the added generalization that allowed the particle to have radial or z-axis motion. Really that was no longer a circular motion problem, but really just a calculation where I was playing with the use of cylindrical coordinates to describe the motion.

It occurred to me that this can be done without any use of Geometric Algebra (or Pauli matrices), which is probably how I should have attempted it on the exam. Let's use a hybrid coordinate vector and complex number representation to describe the particle position

$$
\mathbf{x}_c = \begin{bmatrix} ae^{i\theta} \\ h \end{bmatrix}, \tag{37}
$$

with the field measurement position of

$$
\mathbf{r} = \begin{bmatrix} \rho e^{i\phi} \\ z \end{bmatrix}. \tag{38}
$$

The particle velocity is

$$
\mathbf{v}_c = \begin{bmatrix} (\dot{a} + ia\dot{\theta})e^{i\theta} \\ \dot{h} \end{bmatrix} = \begin{bmatrix} e^{i\theta} & ie^{i\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{a} \\ a\dot{\theta} \\ \dot{h} \end{bmatrix} \tag{39}
$$

We also want the vectorial difference between the field measurement position and the particle position

$$\mathbf{R} = \mathbf{r} - \mathbf{x}_c = \begin{bmatrix} e^{i\phi} & -e^{i\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho \\ a \\ z-h \end{bmatrix}. \tag{40}$$

The dot product between $\mathbf{R}$ and $\mathbf{v}_c$ is then

$$\begin{aligned}
\mathbf{v}_c \cdot \mathbf{R} &= \begin{bmatrix} \dot{a} & a\dot{\theta} & \dot{h} \end{bmatrix} \operatorname{Re} \left(\begin{bmatrix} e^{-i\theta} & 0 \\ -ie^{-i\theta} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{i\phi} & -e^{i\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} \rho \\ a \\ z-h \end{bmatrix} \\
&= \begin{bmatrix} \dot{a} & a\dot{\theta} & \dot{h} \end{bmatrix} \operatorname{Re} \left(\begin{bmatrix} e^{i(\phi-\theta)} & -1 & 0 \\ -ie^{i(\phi-\theta)} & i & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} \rho \\ a \\ z-h \end{bmatrix} \\
&= \begin{bmatrix} \dot{a} & a\dot{\theta} & \dot{h} \end{bmatrix} \begin{bmatrix} \cos(\phi-\theta) & -1 & 0 \\ \sin(\phi-\theta) & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho \\ a \\ z-h \end{bmatrix}.
\end{aligned}$$

Expansion of the final matrix products is then

$$\mathbf{v}_c \cdot \mathbf{R} = \dot{h}(z-h) - a\dot{a} + \rho\dot{a}\cos(\phi-\theta) + \rho a^2\dot{\theta}\sin(\phi-\theta) \tag{41}$$

The other quantity that we want is $\mathbf{R}^2$, which is

$$\begin{aligned}
\mathbf{R}^2 &= \begin{bmatrix} \rho & a & (z-h) \end{bmatrix} \operatorname{Re} \left(\begin{bmatrix} e^{-i\phi} & 0 \\ -e^{-i\theta} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{i\phi} & -e^{i\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} \rho \\ a \\ z-h \end{bmatrix} \\
&= \begin{bmatrix} \rho & a & (z-h) \end{bmatrix} \begin{bmatrix} 1 & -\cos(\phi-\theta) & 0 \\ -\cos(\phi-\theta) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho \\ a \\ z-h \end{bmatrix}
\end{aligned}$$

The retarded time at which the field is measured is therefore defined implicitly by

$$R = \sqrt{(\rho^2 + (a(t_r))^2 + (z-h(t_r))^2 - 2a(t_r)\rho\cos(\phi-\theta(t_r))} = c(t-t_r). \tag{42}$$

Together 39, 41, and 42 define the four potentials

$$\begin{aligned}
A^0 &= \frac{q}{R - \mathbf{R}\cdot\mathbf{v}_c/c} \qquad (43) \\
\mathbf{A} &= \frac{\mathbf{v}_c}{c} A^0, \qquad (44)
\end{aligned}$$

where all quantities are evaluated at the retarded time t_r given by 42.
In the homework (and in the text [2] §63) we found for $\mathbf{E}$ and $\mathbf{B}$

$$\mathbf{E} = e(1-\beta_c^2)\frac{\hat{\mathbf{R}} - \boldsymbol{\beta}_c}{R^2(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}_c)^3} + e\frac{1}{R(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}_c)^3}\hat{\mathbf{R}} \times ((\hat{\mathbf{R}} - \boldsymbol{\beta}_c) \times \mathbf{a}_c/c^2) \tag{45}$$

$$\mathbf{B} = \hat{\mathbf{R}} \times \mathbf{E}. \tag{46}$$

Expanding out the cross products this yields

$$\mathbf{E} = e(1-\beta_c^2)\frac{\hat{\mathbf{R}} - \boldsymbol{\beta}_c}{R^2(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}_c)^3} + e\frac{1}{R(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}_c)^3}(\hat{\mathbf{R}} - \boldsymbol{\beta}_c)\left(\hat{\mathbf{R}}\cdot\frac{\mathbf{a}_c}{c^2}\right) - e\frac{1}{R(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}_c)^2}\frac{\mathbf{a}_c}{c^2} \tag{47}$$

$$\mathbf{B} = e(1-\beta_c^2)\frac{\boldsymbol{\beta}_c \times \hat{\mathbf{R}}}{R^2(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}_c)^3} + e\frac{1}{R(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}_c)^3}(\boldsymbol{\beta}_c \times \hat{\mathbf{R}})\left(\hat{\mathbf{R}}\cdot\frac{\mathbf{a}_c}{c^2}\right) + e\frac{1}{R(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}_c)^2}\frac{\mathbf{a}_c}{c^2} \times \hat{\mathbf{R}} \tag{48}$$

While longer, it is nice to call out the symmetry between $\mathbf{E}$ and $\mathbf{B}$ explicitly. As a side note, how do these combine in the Geometric Algebra formalism where we have $F = \mathbf{E} + I\mathbf{B}$? That gives us

$$F = e\frac{1}{(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}_c)^3}\left(\left(\frac{1-\beta_c^2}{R^2} + \frac{\hat{\mathbf{R}}\cdot\mathbf{a}_c}{cR}\right)(\hat{\mathbf{R}} - \boldsymbol{\beta}_c + \hat{\mathbf{R}} \wedge (\hat{\mathbf{R}} - \boldsymbol{\beta}_c)) + \frac{1}{R}\left(\frac{\mathbf{a}_c}{c^2} + \frac{\mathbf{a}_c}{c^2} \wedge \hat{\mathbf{R}}\right)\right) \tag{49}$$

I'd guess a multivector of the form $\mathbf{a} + \mathbf{a} \wedge \hat{\mathbf{b}}$, can be tidied up a bit more, but this won't be persued here. Instead let's write out the fields corresponding to the potentials of 43 explicitly. We need to calculate $\mathbf{a}_c$, $\mathbf{v}_c \times \mathbf{R}$, $\mathbf{a}_c \times \mathbf{R}$, and $\mathbf{a}_c \cdot \mathbf{R}$. For the acceleration we get

$$\mathbf{a}_c = \begin{bmatrix} \left(\ddot{a} - a\dot{\theta}^2 + i(a\ddot{\theta} + 2\dot{a}\dot{\theta})\right) e^{i\theta} \\ \ddot{h} \end{bmatrix} \tag{50}$$

Dotted with $\mathbf{R}$ we have

$$\begin{aligned}
\mathbf{a}_c \cdot \mathbf{R} &= \begin{bmatrix} \left(\ddot{a} - a\dot{\theta}^2 + i(a\ddot{\theta} + 2\dot{a}\dot{\theta})\right) e^{i\theta} \\ \ddot{h} \end{bmatrix} \cdot \begin{bmatrix} \rho e^{i\phi} - a e^{i\theta} \\ h \end{bmatrix} \\
&= h\ddot{h} + \mathrm{Re}\left(\left(\ddot{a} - a\dot{\theta}^2 + i(a\ddot{\theta} + 2\dot{a}\dot{\theta})\right)\left(\rho e^{i(\theta-\phi)} - a\right)\right),
\end{aligned}$$

which gives us

$$\mathbf{a}_c \cdot \mathbf{R} = h\ddot{h} + (\ddot{a} - a\dot{\theta}^2)(\rho\cos(\phi - \theta) - a) + (a\ddot{\theta} + 2\dot{a}\dot{\theta})\rho\sin(\phi - \theta). \tag{51}$$

Now, how do we handle the cross products in this complex number, scalar hybrid format? With some playing around such a cross product can be put into the following tidy form

$$\begin{bmatrix} z_1 \\ h_1 \end{bmatrix} \times \begin{bmatrix} z_2 \\ h_2 \end{bmatrix} = \begin{bmatrix} i(h_1 z_2 - h_2 z_1) \\ \mathrm{Im}(z_1^* z_2) \end{bmatrix}. \tag{52}$$

This is a sensible result. Crossing with $\mathbf{e}_3$ will rotate in the $x-y$ plane, which accounts for the factors of i in the complex portion of the cross product. The imaginary part has only contributions from the portions of the vectors z_1 and z_2 that are perpendicular to each other, so while the real part of $z_1^* z_2$ measures the colinearity, the imaginary part is a measure of the amount perpendicular.

Using this for our velocity cross product we have

$$\begin{aligned}
\mathbf{v}_c \times \mathbf{R} &= \begin{bmatrix} (\dot{a} + ia\dot{\theta})e^{i\theta} \\ \dot{h} \end{bmatrix} \times \begin{bmatrix} \rho e^{i\phi} - a e^{i\theta} \\ h \end{bmatrix} \\
&= \begin{bmatrix} i\left(\dot{h}(\rho e^{i\phi} - a e^{i\theta}) - h(\dot{a} + ia\dot{\theta})e^{i\theta}\right) \\ \operatorname{Im}\Big((\dot{a} - ia\dot{\theta})(\rho e^{i(\phi-\theta)} - a)\Big) \end{bmatrix}
\end{aligned}$$

which is

$$\mathbf{v}_c \times \mathbf{R} = \begin{bmatrix} i(\dot{h}\rho e^{i\phi} - (h\dot{a} + iha\dot{\theta} + a\dot{h})e^{i\theta}) \\ \dot{a}\rho\sin(\phi-\theta) - a\dot{\theta}\rho\cos(\phi-\theta) + a^2\dot{\theta} \end{bmatrix}. \tag{53}$$

The last thing required to write out the fields is

$$\begin{aligned}
\mathbf{a}_c \times \mathbf{R} &= \begin{bmatrix} \left(\ddot{a} - a\dot{\theta}^2 + i(a\ddot{\theta} + 2\dot{a}\dot{\theta})\right) e^{i\theta} \\ \ddot{h} \end{bmatrix} \times \begin{bmatrix} \rho e^{i\phi} - a e^{i\theta} \\ z - h \end{bmatrix} \\
&= \begin{bmatrix} i\ddot{h}(\rho e^{i\phi} - a e^{i\theta}) - i(z-h)\left(\ddot{a} - a\dot{\theta}^2 + i(a\ddot{\theta} + 2\dot{a}\dot{\theta})\right) e^{i\theta} \\ \operatorname{Im}\Big(\left(\ddot{a} - a\dot{\theta}^2 - i(a\ddot{\theta} + 2\dot{a}\dot{\theta})\right)(\rho e^{i(\phi-\theta)} - a)\Big) \end{bmatrix}
\end{aligned}$$

So the acceleration cross product is

$$\mathbf{a}_c \times \mathbf{R} = \begin{bmatrix} i\ddot{h}\rho e^{i\phi} - i\left(\ddot{h}a + (z-h)\left(\ddot{a} - a\dot{\theta}^2 + i(a\ddot{\theta} + 2\dot{a}\dot{\theta})\right)\right) e^{i\theta} \\ \left(\ddot{a} - a\dot{\theta}^2\right)\rho\sin(\phi-\theta) - (a\ddot{\theta} + 2\dot{a}\dot{\theta})(\rho\cos(\phi-\theta) - a) \end{bmatrix} \tag{54}$$

Putting all the results together creates something that is too long to easily write, but can at least be summarized

$$\mathbf{E} = \frac{e}{(R - \mathbf{R}\cdot\boldsymbol{\beta}_c)^3}\left(\left(1 - \beta_c^2 + \mathbf{R}\cdot\frac{\mathbf{a}_c}{c^2}\right)(\mathbf{R} - \boldsymbol{\beta}_c R) - R(R - \mathbf{R}\cdot\boldsymbol{\beta}_c)\frac{\mathbf{a}_c}{c^2}\right) \tag{55}$$

$$\mathbf{B} = \frac{e}{(R - \mathbf{R}\cdot\boldsymbol{\beta}_c)^3}\left(\left(1 - \beta_c^2 + \mathbf{R}\cdot\frac{\mathbf{a}_c}{c^2}\right)(\boldsymbol{\beta}_c \times \mathbf{R}) - (R - \mathbf{R}\cdot\boldsymbol{\beta}_c)\frac{\mathbf{a}_c}{c^2}\times\mathbf{R}\right) \tag{56}$$

$$1 - \beta_c^2 = 1 - (\dot{a}^2 + a^2\dot{\theta}^2 + \dot{h}^2)/c^2 \tag{57}$$

$$R = \sqrt{(\rho^2 + (a(t_r))^2 + (z - h(t_r))^2 - 2a(t_r)\rho\cos(\phi - \theta(t_r))} = c(t - t_r) \tag{58}$$

$$\mathbf{R} - \boldsymbol{\beta}_c R = \begin{bmatrix} \rho e^{i\phi} - (a + (\dot{a} + ia\dot{\theta})R/c)e^{i\theta} \\ z - h - \dot{h}R/c \end{bmatrix} \tag{59}$$

$$\boldsymbol{\beta}_c \cdot \mathbf{R} = \frac{1}{c}\left(\dot{h}(z-h) - a\dot{a} + \rho\dot{a}\cos(\phi - \theta) + \rho a^2\dot{\theta}\sin(\phi - \theta)\right) \tag{60}$$

$$\boldsymbol{\beta}_c \times \mathbf{R} = \frac{1}{c}\begin{bmatrix} i(\dot{h}\rho e^{i\phi} - (h\dot{a} + iha\dot{\theta} + a\dot{h})e^{i\theta}) \\ \dot{a}\rho\sin(\phi - \theta) - a\dot{\theta}\rho\cos(\phi - \theta) + a^2\dot{\theta} \end{bmatrix} \tag{61}$$

$$\frac{\mathbf{a}_c}{c^2} = \frac{1}{c^2}\begin{bmatrix} \left(\ddot{a} - a\dot{\theta}^2 + i(a\ddot{\theta} + 2\dot{a}\dot{\theta})\right)e^{i\theta} \\ \ddot{h} \end{bmatrix} \tag{62}$$

$$\frac{\mathbf{a}_c}{c^2}\cdot\mathbf{R} = \frac{1}{c^2}\left(h\ddot{h} + (\ddot{a} - a\dot{\theta}^2)(\rho\cos(\phi - \theta) - a) + (a\ddot{\theta} + 2\dot{a}\dot{\theta})\rho\sin(\phi - \theta)\right) \tag{63}$$

$$\frac{\mathbf{a}_c}{c^2}\times\mathbf{R} = \frac{1}{c^2}\begin{bmatrix} i\ddot{h}\rho e^{i\phi} - i\left(\ddot{h}a + (z-h)\left(\ddot{a} - a\dot{\theta}^2 + i(a\ddot{\theta} + 2\dot{a}\dot{\theta})\right)\right)e^{i\theta} \\ \left(\ddot{a} - a\dot{\theta}^2\right)\rho\sin(\phi - \theta) - (a\ddot{\theta} + 2\dot{a}\dot{\theta})(\rho\cos(\phi - \theta) - a) \end{bmatrix}. \tag{64}$$

This is a whole lot more than the exam question asked for, since it is actually the most general solution to the electric and magnetic fields associated with an arbitrary charged particle (when that motion is described in cylindrical coordinates). The exam question had $\theta = kct$ and $\dot{a} = 0, h = 0$, which kills a number of the terms

$$1 - \beta_c^2 + \frac{\mathbf{a}_c}{c^2}\cdot\mathbf{R} = 1 - ak^2\rho\cos(\phi - kct_r) \tag{65}$$

$$R = \sqrt{(\rho^2 + a^2 + z^2 - 2a\rho\cos(\phi - kct_r)} = c(t - t_r) \tag{66}$$

$$\mathbf{R} - \boldsymbol{\beta}_c R = \begin{bmatrix} \rho e^{i\phi} - a(1 + ikR)e^{ikct_r} \\ z \end{bmatrix} \tag{67}$$

$$\boldsymbol{\beta}_c \cdot \mathbf{R} = \rho a^2 k\sin(\phi - kct_r) \tag{68}$$

$$\boldsymbol{\beta}_c \times \mathbf{R} = \begin{bmatrix} 0 \\ ak(a - \rho\cos(\phi - kct_r)) \end{bmatrix} \tag{69}$$

$$\frac{\mathbf{a}_c}{c^2} = \begin{bmatrix} -ak^2 e^{ikct_r} \\ 0 \end{bmatrix} \tag{70}$$

$$\frac{\mathbf{a}_c}{c^2}\times\mathbf{R} = \begin{bmatrix} iza k^2 e^{ikct_r} \\ -ak^2\rho\sin(\phi - kct_r) \end{bmatrix}. \tag{71}$$

This is still messy, but is a satisfactory solution to the problem.

The exam question also asked only about the $\rho = 0$, so ϕ also becomes irrelevant. In that case we have along the z-axis the fields are given by

$$\mathbf{E}(z) = \frac{e}{R^3} \begin{bmatrix} -a(1 + ikR - k^2R^2)e^{ik(ct-R)} \\ z \end{bmatrix} \tag{72}$$

$$\mathbf{B}(z) = \frac{e}{R^3} \begin{bmatrix} -Rizak^2e^{ik(ct-R)} \\ a^2k \end{bmatrix} \tag{73}$$

$$R = \sqrt{a^2 + z^2} \tag{74}$$

Similar to when things were calculated from the potentials directly, I get a different result from $\hat{\mathbf{R}} \times \mathbf{E}$

$$\hat{\mathbf{R}} \times \mathbf{E}(z) = \frac{e}{R^3} \begin{bmatrix} akz(1 + ikR)e^{ik(ct-R)} \\ -a^2k \end{bmatrix} \tag{75}$$

compared to the value of $\mathbf{B}$ that was directly calculated above. With the sign swapped in the z-axis term of $\mathbf{B}(z)$ here I'd guess I've got an algebraic error hiding somewhere?

2. **Collision of photon and electron.**

I made a dumb error on the exam on this one. I setup the four momentum conservation statement, but then didn't multiply out the cross terms properly. This led me to incorrectly assume that I had to try doing this the hard way (something akin to what I did on the midterm). Simon later told us in the tutorial the simple way, and that's all we needed here too. Here's the setup.

An electron at rest initially has four momentum

$$(mc, 0) \tag{76}$$

where the incoming photon has four momentum

$$\left(\hbar\frac{\omega}{c}, \hbar\mathbf{k}\right) \tag{77}$$

After the collision our electron has some velocity so its four momentum becomes (say)

$$\gamma(mc, m\mathbf{v}), \tag{78}$$

and our new photon, going off on an angle θ relative to $\mathbf{k}$ has four momentum

$$\left(\hbar\frac{\omega'}{c}, \hbar\mathbf{k}'\right) \tag{79}$$

Our conservation relationship is thus

$$(mc, 0) + \left(\hbar\frac{\omega}{c}, \hbar\mathbf{k}\right) = \gamma(mc, m\mathbf{v}) + \left(\hbar\frac{\omega'}{c}, \hbar\mathbf{k}'\right) \tag{80}$$

I squared both sides, but dropped my cross terms, which was just plain wrong, and costly for both time and effort on the exam. What I should have done was just

$$\gamma(mc, m\mathbf{v}) = (mc, 0) + \left(\hbar\frac{\omega}{c}, \hbar\mathbf{k}\right) - \left(\hbar\frac{\omega'}{c}, \hbar\mathbf{k}'\right), \tag{81}$$

and then square this (really making contractions of the form $p_i p^i$). That gives (and this time keeping my cross terms)

$$
\begin{aligned}
(\gamma(mc, m\mathbf{v}))^2 &= \gamma^2 m^2 (c^2 - \mathbf{v}^2) \\
&= m^2c^2 \\
&= m^2c^2 + 0 + 0 + 2(mc, 0) \cdot \left(\hbar\frac{\omega}{c}, \hbar\mathbf{k}\right) - 2(mc, 0) \cdot \left(\hbar\frac{\omega'}{c}, \hbar\mathbf{k}'\right) - 2 \cdot \left(\hbar\frac{\omega}{c}, \hbar\mathbf{k}\right) \cdot \left(\hbar\frac{\omega'}{c}, \hbar\mathbf{k}'\right) \\
&= m^2c^2 + 2mc\hbar\frac{\omega}{c} - 2mc\hbar\frac{\omega'}{c} - 2\hbar^2 \left(\frac{\omega}{c}\frac{\omega'}{c} - \mathbf{k} \cdot \mathbf{k}'\right) \\
&= m^2c^2 + 2mc\hbar\frac{\omega}{c} - 2mc\hbar\frac{\omega'}{c} - 2\hbar^2 \frac{\omega}{c}\frac{\omega'}{c}(1 - \cos\theta)
\end{aligned}
$$

Rearranging a bit we have

$$\omega' \left(m + \frac{\hbar\omega}{c^2}(1 - \cos\theta)\right) = m\omega, \tag{82}$$

or

$$\omega' = \frac{\omega}{1 + \frac{\hbar\omega}{mc^2}(1 - \cos\theta)} \tag{83}$$

3. **Pion decay.**

The problem above is very much like a midterm problem we had, so there was no justifiable excuse for messing up on it. That midterm problem was to consider the split of a pion at rest into a neutrino (massless) and a muon, and to calculate the energy of the muon. That one also follows the same pattern, a calculation of four momentum conservation, say

$$(m_\pi c, 0) = \hbar\frac{\omega}{c}(1, \hat{\mathbf{k}}) + (\mathcal{E}_\mu/c, \mathbf{p}_\mu). \tag{84}$$

Here ω is the frequency of the massless neutrino. The massless nature is encoded by a four momentum that squares to zero, which follows from $(1, \hat{\mathbf{k}}) \cdot (1, \hat{\mathbf{k}}) = 1^2 - \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 0$.

When I did this problem on the midterm, I perversely put in a scattering angle, instead of recognizing that the particles must scatter at 180 degree directions since spatial momentum components must also be preserved. This and the combination of trying to work in spatial quantities led to a mess and I didn't get the end result in anything that could be considered tidy.

The simple way to do this is to just rearrange to put the null vector on one side, and then square. This gives us

$$
\begin{aligned}
0 &= \left(\hbar\frac{\omega}{c}(1, \hat{\mathbf{k}})\right) \cdot \left(\hbar\frac{\omega}{c}(1, \hat{\mathbf{k}})\right) \\
&= \big((m_\pi c, 0) - (\mathcal{E}_\mu/c, \mathbf{p}_\mu)\big) \cdot \big((m_\pi c, 0) - (\mathcal{E}_\mu/c, \mathbf{p}_\mu)\big) \\
&= {m_\pi}^2c^2 + {m_\nu}^2c^2 - 2(m_\pi c, 0) \cdot (\mathcal{E}_\mu/c, \mathbf{p}_\mu) \\
&= {m_\pi}^2c^2 + {m_\nu}^2c^2 - 2m_\pi\mathcal{E}_\mu
\end{aligned}
$$

A final re-arrangement gives us the muon energy

$$\mathcal{E}_\mu = \frac{1}{2}\frac{{m_\pi}^2 + {m_\nu}^2}{m_\pi} c^2 \tag{85}$$

References

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