PHY450H1S. Relativistic Electrodynamics Lecture 10 (Taught by Prof. Erich Poppitz). Lorentz force equation energy term, and four vector formulation of the Lorentz force equation.

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1. Reading.

Covering chapter 3 material from the text [1].

Covering lecture notes pp. 74-83: gauge transformations in 3-vector language (74); energy of a relativistic particle in EM field (75); variational principle and equation of motion in 4-vector form (76-77); the field strength tensor (78-80); the fourth equation of motion (81)

2. What is the significance to the gauge invariance of the action?

We had argued that under a gauge transformation

$$A_i \rightarrow A_i + \frac{\partial \chi}{\partial x^i}, \tag{1}$$

the action for a particle changes by a boundary term

$$-\frac{e}{c}(\chi(x_b) - \chi(x_a)). \tag{2}$$

Because S changes by a boundary term only, variation problem is not affected. The extremal trajectories are then the same, hence the EOM are the same.

2.1. *A less high brow demonstration.*

With our four potential split into space and time components

$$A^i = (\phi, \mathbf{A}), \tag{3}$$

the lower index representation of the same vector is

$$A_i = (\phi, -\mathbf{A}). \tag{4}$$

Our gauge transformation is then

$$A_0 \rightarrow A_0 + \frac{\partial \chi}{\partial x^0} \tag{5}$$

$$-\mathbf{A} \rightarrow -\mathbf{A} + \frac{\partial \chi}{\partial \mathbf{x}} \tag{6}$$

or

$$\phi \to \phi + \frac{1}{c}\frac{\partial \chi}{\partial t} \tag{7}$$

$$\mathbf{A} \to \mathbf{A} - \boldsymbol{\nabla}\chi. \tag{8}$$

Now observe how the electric and magnetic fields are transformed

$$\begin{aligned}
\mathbf{E} &= -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \\
&\to -\boldsymbol{\nabla}\left(\phi + \frac{1}{c}\frac{\partial \chi}{\partial t}\right) - \frac{1}{c}\frac{\partial}{\partial t}\left(\mathbf{A} - \boldsymbol{\nabla}\chi\right)
\end{aligned}$$

Sufficient continuity of χ is assumed, allowing commutation of the space and time derivatives, and we are left with just $\mathbf{E}$

For the magnetic field we have

$$\begin{aligned}
\mathbf{B} &= \boldsymbol{\nabla} \times \mathbf{A} \\
&\to \boldsymbol{\nabla} \times (\mathbf{A} - \boldsymbol{\nabla}\chi)
\end{aligned}$$

Again with continuity assumptions, $\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\chi) = 0$, and we are left with just $\mathbf{B}$. The electromagnetic fields (as opposed to potentials) do not change under gauge transformations.

We conclude that the $\{A_i\}$ description is hugely redundant, but despite that, local $\mathcal{L}$ and H can only be written in terms of the potentials A_i.

2.2. *Energy term of the Lorentz force. Three vector approach.*

With the Lagrangian for the particle given by

$$\mathcal{L} = -mc^2\sqrt{1 - \frac{\mathbf{v}^2}{c^2}} + \frac{e}{c}\mathbf{A}\cdot\mathbf{v} - e\phi, \tag{9}$$

we define the energy as

$$\mathcal{E} = \mathbf{v}\cdot\frac{\partial \mathcal{L}}{\partial \mathbf{v}} - \mathcal{L} \tag{10}$$

This is not necessarily a conserved quantity, but we define it as the energy anyways (we don't really have a Hamiltonian when the fields are time dependent). Associated with this quantity is the general relationship

$$\frac{d\mathcal{E}}{dt} = -\frac{\partial \mathcal{L}}{\partial t}, \tag{11}$$

and when the Lagrangian is invariant with respect to time translation the energy $\mathcal{E}$ will be a conserved quantity (and also the Hamiltonian).

Our canonical momentum is

$$\frac{\partial \mathcal{L}}{\partial \mathbf{v}} = \gamma m\mathbf{v} + \frac{e}{c}\mathbf{A} \tag{12}$$

So our energy is

$$\mathcal{E} = \gamma m \mathbf{v}^2 + \frac{e}{c}\mathbf{A}\cdot\mathbf{v} - \left(-mc^2\sqrt{1-\frac{\mathbf{v}^2}{c^2}} + \frac{e}{c}\mathbf{A}\cdot\mathbf{v} - e\phi\right).$$

Or

$$\mathcal{E} = \underbrace{\frac{mc^2}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}}_{(*)} + e\phi. \tag{13}$$

The contribution of $(*)$ to the energy $\mathcal{E}$ comes from the free (kinetic) particle portion of the Lagrangian $\mathcal{L} = -mc^2\sqrt{1-\frac{\mathbf{v}^2}{c^2}}$, and we identify the remainder as a potential energy

$$\mathcal{E} = \frac{mc^2}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}} + \underbrace{e\phi}_{\text{"potential"}}. \tag{14}$$

For the kinetic portion we can also show that we have

$$\frac{d}{dt}\mathcal{E}_{\text{kinetic}} = \frac{mc^2}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}} = e\mathbf{E}\cdot\mathbf{v}. \tag{15}$$

To show this observe that we have

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_{\text{kinetic}} &= mc^2\frac{d\gamma}{dt} \\
&= mc^2\frac{d}{dt}\frac{1}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}} \\
&= mc^2\frac{\frac{\mathbf{v}}{c^2}\cdot\frac{d\mathbf{v}}{dt}}{\left(1-\frac{\mathbf{v}^2}{c^2}\right)^{3/2}} \\
&= \frac{m\gamma\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}}{1-\frac{\mathbf{v}^2}{c^2}}
\end{aligned}$$

We also have

$$\begin{aligned}
\mathbf{v}\cdot\frac{d\mathbf{p}}{dt} &= \mathbf{v}\cdot\frac{d}{dt}\frac{m\mathbf{v}}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}} \\
&= m\mathbf{v}^2\frac{d\gamma}{dt} + m\gamma\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} \\
&= m\mathbf{v}^2\frac{d\gamma}{dt} + mc^2\frac{d\gamma}{dt}\left(1-\frac{\mathbf{v}^2}{c^2}\right) \\
&= mc^2\frac{d\gamma}{dt}.
\end{aligned}$$

Utilizing the Lorentz force equation, we have

$$\mathbf{v} \cdot \frac{d\mathbf{p}}{dt} = e\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \cdot \mathbf{v} = e\mathbf{E} \cdot \mathbf{v} \tag{16}$$

and are able to assemble the above, and find that we have

$$\frac{d(mc^2\gamma)}{dt} = e\mathbf{E} \cdot \mathbf{v} \tag{17}$$

3. **Four vector Lorentz force**

Using $ds = \sqrt{dx^i dx_i}$ our action can be rewritten

$$\begin{aligned}
S &= \int \left(-mcds - \frac{e}{c}u^i A_i ds\right) \\
&= \int \left(-mcds - \frac{e}{c}dx^i A_i\right) \\
&= \int \left(-mc\sqrt{dx^i dx_i} - \frac{e}{c}dx^i A_i\right)
\end{aligned}$$

$x^i(\tau)$ is a worldline $x^i(0) = a^i$, $x^i(1) = b^i$,
We want $\delta S = S[x + \delta x] - S[x] = 0$ (to linear order in δx)
The variation of our proper length is

$$\begin{aligned}
\delta ds &= \delta\sqrt{dx^i dx_i} \\
&= \frac{1}{2\sqrt{dx^i dx_i}}\delta(dx^j dx_j)
\end{aligned}$$

Observe that for the numerator we have

$$\begin{aligned}
\delta(dx^j dx_j) &= \delta(dx^j g_{jk} dx^k) \\
&= \delta(dx^j) g_{jk} dx^k + dx^j g_{jk} \delta(dx^k) \\
&= \delta(dx^j) g_{jk} dx^k + dx^k g_{kj} \delta(dx^j) \\
&= 2\delta(dx^j) g_{jk} dx^k \\
&= 2\delta(dx^j) dx_j
\end{aligned}$$

TIP: If this goes too quick, or there is any disbelief, write these all out explicitly as $dx^j dx_j = dx^0 dx_0 + dx^1 dx_1 + dx^2 dx_2 + dx^3 dx_3$ and compute it that way.
For the four vector potential our variation is

$$\delta A_i = A_i(x + \delta x) - A_i = \frac{\partial A_i}{\partial x^j}\delta x^j = \partial_j A_i \delta x^j \tag{18}$$

(i.e. By chain rule)
Completing the proper length variations above we have

$$
\begin{aligned}
\delta\sqrt{dx^i dx_i} &= \frac{1}{\sqrt{dx^i dx_i}}\delta(dx^j)dx_j \\
&= \delta(dx^j)\frac{dx_j}{ds} \\
&= \delta(dx^j)u_j \\
&= d\delta x^j u_j
\end{aligned}
$$

We are now ready to assemble results and do the integration by parts

$$
\begin{aligned}
\delta S &= \int \left(-mcd(\delta x^j)u_j - \frac{e}{c}d(\delta x^i)A_i - \frac{e}{c}dx^i\partial_j A_i \delta x^j\right) \\
&= \left(-mc(\delta x^j)u_j - \frac{e}{c}(\delta x^i)A_i\right)\Big|_a^b + \int \left(mc\delta x^j du_j + \frac{e}{c}(\delta x^i)dA_i - \frac{e}{c}dx^i\partial_j A_i \delta x^j\right)
\end{aligned}
$$

Our variation at the endpoints is zero $\delta x^i\big|_a = \delta x^i\big|_b = 0$, killing the non-integral terms

$$
\delta S = \int \delta x^j \left(mcdu_j + \frac{e}{c}dA_j - \frac{e}{c}dx^i\partial_j A_i\right).
$$

Observe that our differential can also be expanded by chain rule

$$
dA_j = \frac{\partial A_j}{\partial x^i}dx^i = \partial_i A_j dx^i, \tag{19}
$$

which simplifies the variation further

$$
\begin{aligned}
\delta S &= \int \delta x^j \left(mcdu_j + \frac{e}{c}dx^i(\partial_i A_j - \partial_j A_i)\right) \\
&= \int \delta x^j ds \left(mc\frac{du_j}{ds} + \frac{e}{c}u^i(\partial_i A_j - \partial_j A_i)\right)
\end{aligned}
$$

Since this is true for all variations δx^j, which is arbitrary, the interior part is zero everywhere in the trajectory. The antisymmetric portion, a rank 2 4-tensor is called the electromagnetic field strength tensor, and written

$$
\boxed{F_{ij} = \partial_i A_j - \partial_j A_i.} \tag{20}
$$

In matrix form this is

$$
\|F_{ij}\| = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0. \end{bmatrix} \tag{21}
$$

In terms of the field strength tensor our Lorentz force equation takes the form

$$\boxed{\frac{d(mcu_i)}{ds} = \frac{e}{c} F_{ij} u^j.} \tag{22}$$

References

[1] L.D. Landau and E.M. Lifshits. *The classical theory of fields*. Butterworth-Heinemann, 1980. 1