PHY450H1S. Relativistic Electrodynamics Lecture 19 (Taught by Prof. Erich Poppitz). Lienard-Wiechert potentials.

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Peeter Joot — peeter.joot@gmail.com
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1. **Reading.**

Covering chapter 8 material from the text [1].

Covering lecture notes pp. 136-146: the Lienard-Wiechert potentials (143-146) [Wednesday, Mar. 9...]

2. **Fields from the Lienard-Wiechert potentials**

(We finished off with the scalar and vector potentials in class, but I've put those notes with the previous lecture).

To find $\mathbf{E}$ and $\mathbf{B}$ need
$\frac{\partial t_r}{\partial t}$, and $\boldsymbol{\nabla} t_r(\mathbf{x}, t)$
where

$$t - t_r = \left\lvert \mathbf{x} - \mathbf{x}_c(t_r) \right\rvert \tag{1}$$

implicit definition of $t_r(\mathbf{x}, t)$
In HW5 you'll show

$$\frac{\partial t_r}{\partial t} = \frac{\left\lvert \mathbf{x} - \mathbf{x}_c(t_r) \right\rvert}{\left\lvert \mathbf{x} - \mathbf{x}_c(t_r) \right\rvert - \frac{\mathbf{v}_c}{c} \cdot (\mathbf{x} - \mathbf{x}_c(t_r))} \tag{2}$$

$$\boldsymbol{\nabla} t_r = \frac{1}{c} \frac{\mathbf{x} - \mathbf{x}_c(t_r)}{\left\lvert \mathbf{x} - \mathbf{x}_c(t_r) \right\rvert - \frac{\mathbf{v}_c}{c} \cdot (\mathbf{x} - \mathbf{x}_c(t_r))} \tag{3}$$

and then use this to show that the electric and magnetic fields due to a moving charge are

$$\begin{aligned}
\mathbf{E}(\mathbf{x}, t) &= \frac{eR}{(\mathbf{R} \cdot \mathbf{u})^3} \left((c^2 - \mathbf{v}_c^2)\mathbf{u} + \mathbf{R} \times (\mathbf{u} \times \mathbf{a}_c) \right) && (4) \\
&= \frac{\mathbf{R}}{R} \times \mathbf{E} && (5) \\
\mathbf{u} &= c\frac{\mathbf{R}}{R} - \mathbf{v}_c, && (6)
\end{aligned}$$

where everything is evaluated at the retarded time $t_r = t - \left\lvert \mathbf{x} - \mathbf{x}_c(t_r) \right\rvert / c$.

This looks quite a bit different than what we find in §63 (63.8) in the text, but a little bit of expansion shows they are the same.

3. **Check. Particle at rest.**

With

$$\begin{aligned}
\mathbf{x}_c &= \mathbf{x}_0 \\
X_c^k &= (ct, \mathbf{x}_0) \\
|\mathbf{x} - \mathbf{x}_c(t_r)| &= c(t - t_r)
\end{aligned}$$

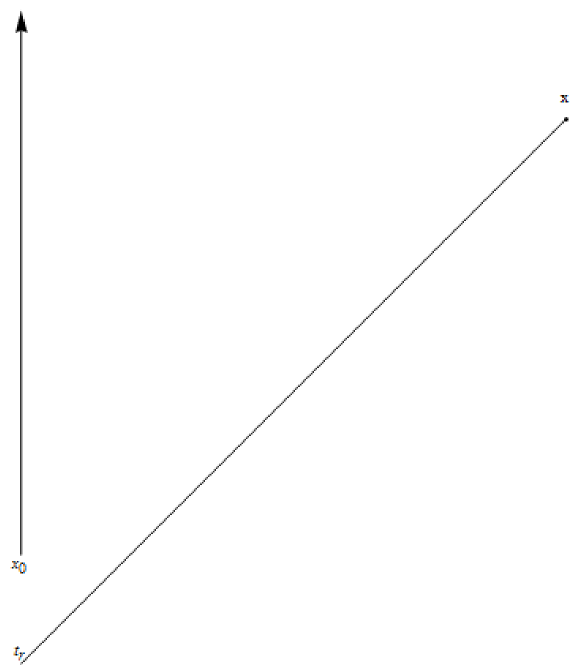

Figure 1: Retarded time for particle at rest.

As illustrated in figure (1) the retarded position is

$$\mathbf{x}_c(t_r) = \mathbf{x}_0, \tag{7}$$

for

$$\mathbf{u} = \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|} c, \tag{8}$$

and

$$\mathbf{E} = e \frac{\cancel{|\mathbf{x} - \mathbf{x}_0|}}{(c|\mathbf{x} - \mathbf{x}_0|)^3} c^3 \frac{\mathbf{x} - \mathbf{x}_0}{\cancel{|\mathbf{x} - \mathbf{x}_0|}}, \tag{9}$$

which is Coulomb's law

$$\mathbf{E} = e \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|^3} \tag{10}$$

4. **Check. Particle moving with constant velocity.**

This was also computed in full in homework 5. The end result was

$$\mathbf{E} = e\frac{\mathbf{x} - \mathbf{v}t}{\left|\mathbf{x} - \mathbf{v}t\right|^3} \frac{1 - \boldsymbol{\beta}^2}{\left(1 - \frac{(\mathbf{x}\times\boldsymbol{\beta})^2}{\left|\mathbf{x}-\mathbf{v}t\right|^2}\right)^{3/2}} \tag{11}$$

Writing

$$\begin{aligned}
\frac{\mathbf{x}\times\boldsymbol{\beta}}{\left|\mathbf{x} - \mathbf{v}t\right|} &= \frac{1}{c}\frac{(\mathbf{x} - \mathbf{v}t)\times\mathbf{v}}{\left|\mathbf{x} - \mathbf{v}t\right|} \\
&= \frac{\left|\mathbf{v}\right|}{c}\frac{(\mathbf{x} - \mathbf{v}t)\times\mathbf{v}}{\left|\mathbf{x} - \mathbf{v}t\right|\left|\mathbf{v}\right|}
\end{aligned}$$

We can introduce an angular dependence between the charge's translated position and its velocity

$$\sin^2\theta = \left|\frac{\mathbf{v}\times(\mathbf{x} - \mathbf{v}t)}{\left|\mathbf{v}\right|\left|\mathbf{x} - \mathbf{v}t\right|}\right|^2, \tag{12}$$

and write the field as

$$\mathbf{E} = e\underbrace{\frac{\mathbf{x} - \mathbf{v}t}{\left|\mathbf{x} - \mathbf{v}t\right|^3}}_{*} \frac{1 - \boldsymbol{\beta}^2}{\left(1 - \frac{\mathbf{v}^2}{c^2}\sin^2\theta\right)^{3/2}} \tag{13}$$

Observe that $*$ = Coulomb's law measured from the instantaneous position of the charge.

The electric field $\mathbf{E}$ has a time dependence, strongest when perpendicular to the instantaneous position when $\theta = \pi/2$, since the denominator is smallest ($\mathbf{E}$ largest) when $\mathbf{v}/c$ is not small. This is strongly θ dependent.

Compare

$$\begin{aligned}
\frac{\left|\mathbf{E}(\theta = \pi/2)\right| - \left|\mathbf{E}(\theta = \pi/2 + \Delta\theta)\right|}{\left|\mathbf{E}(\theta = \pi/2)\right|} &\approx \frac{\frac{1}{(1-\mathbf{v}^2/c^2)^{3/2}} - \frac{1}{(1-\mathbf{v}^2/c^2(1-(\Delta\theta)^2))^{3/2}}}{\frac{1}{(1-\mathbf{v}^2/c^2)^{3/2}}} \\
&= 1 - \left(\frac{1 - \mathbf{v}^2/c^2}{1 - \mathbf{v}^2/c^2 + \mathbf{v}^2/c^2(\Delta\theta)^2}\right)^{3/2} \\
&= 1 - \left(\frac{1}{1 + \mathbf{v}^2/c^2\frac{(\Delta\theta)^2}{1-\mathbf{v}^2/c^2}}\right)^{3/2}
\end{aligned}$$

Here we used

$$\sin(\theta + \pi/2) = \frac{e^{i(\theta+\pi/2)} - e^{-i(\theta+\pi/2)}}{2i} = \cos\theta \tag{14}$$

and

$$\cos^2 \Delta\theta \approx \left(1 - \frac{(\Delta\theta)^2}{2}\right)^2 \approx 1 - (\Delta\theta)^2 \tag{15}$$

FIXME: he writes:

$$\Delta\theta \le \sqrt{1 - \frac{\mathbf{v}^2}{c^2}} \tag{16}$$

I don't see where that comes from.

FIXME: PICTURE: Various $\mathbf{E}$'s up, and $\mathbf{v}$ perpendicular to that, strongest when charge is moving fast.

5. Back to extracting physics from the Lienard-Wiechert field equations

Imagine that we have a localized particle motion with

$$|\mathbf{x}_c(t_r)| < l \tag{17}$$

The velocity vector

$$\mathbf{u} = c\frac{\mathbf{x} - \mathbf{x}_c(t_r)}{|\mathbf{x} - \mathbf{x}_c|} \tag{18}$$

doesn't grow as distance from the source, so from 4, we have for $|\mathbf{x}| \gg l$

$$\mathbf{B}, \mathbf{E} \sim \frac{1}{|\mathbf{x}|^2}(\cdots) + \frac{1}{\mathbf{x}}(\text{acceleration term}) \tag{19}$$

The acceleration term will dominate at large distances from the source. Our Poynting magnitude is

$$|\mathbf{S}| \sim |\mathbf{E} \times \mathbf{B}| \sim \frac{1}{\mathbf{x}^2}(\text{acceleration})^2. \tag{20}$$

We can ask about

$$\oint d^2\sigma \cdot \mathbf{S} \sim R^2 \frac{1}{R^2}(\text{acceleration})^2 \sim (\text{acceleration})^2 \tag{21}$$

In the limit, for the radiation of EM waves

$$\lim_{R\rightarrow\infty} \oint d^2\sigma \cdot \mathbf{S} \neq 0 \tag{22}$$

The energy flux through a sphere of radius R is called the radiated power.

References

[1] L.D. Landau and E.M. Lifshitz. *The classical theory of fields*. Butterworth-Heinemann, 1980. 1