PHY450H1S. Relativistic Electrodynamics Lecture 24 (Taught by Prof. Erich Poppitz). Non-relativistic electrostatic Lagrangian.

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1. **Reading.**

Covering chapter 5 §37, and chapter 8 §65 material from the text [1].

Covering pp. 181-195: the Lagrangian for a system of non relativistic charged particles to zeroth order in (v/c): electrostatic energy of a system of charges and .mass renormalization.

2. **A closed system of charged particles.**

Consider a closed system of charged particles (m_a, q_a) and imagine there is a frame where they are non-relativistic $v_a/c \ll 1$. In this case we can describe the dynamics using a Lagrangian only for particles. i.e.

$$\mathcal{L} = \mathcal{L}(\mathbf{x}_1, \cdots, \mathbf{x}_N, \mathbf{v}_1, \cdots, \mathbf{v}_N) \tag{1}$$

If we work t order $(v/c)^2$.

If we try to go to $O((v/c)^3$, it's difficult to only use $\mathcal{L}$ for particles.

This can be inferred from

$$P = \frac{2}{3}\frac{e^2}{c^3}\left|\ddot{\mathbf{d}}\right|^2 \tag{2}$$

because at this order, due to radiation effects, we need to include EM field as dynamical.

3. **Start simple**

Start with a system of (non-relativistic) free particles

$$S$$

So in the non-relativistic limit, after dropping the constant term that doesn't effect the dynamics, our Lagrangian is

$$\mathcal{L}(\mathbf{x}_a, \mathbf{v}_a) = \frac{1}{2}\sum_a m_a \mathbf{v}_a^2 - \frac{1}{8}\frac{m_a \mathbf{v}_a^4}{c^2} \tag{3}$$

The first term is $O((v/c)^0)$ where the second is $O((v/c)^2)$.

Next include the fact that particles are charged.

$$\mathcal{L}_{\text{interaction}} = \sum_a \left(\cancel{q_a \frac{\mathbf{v}_a}{c} \cdot \mathbf{A}(\mathbf{x}_a, t)} - q_a \phi(\mathbf{x}_a, t) \right) \tag{4}$$

Here, working to $O((v/c)^0)$, where we consider the particles moving so slowly that we have only a Coulomb potential ϕ, not $\mathbf{A}$.

HERE: these are NOT 'EXTERNAL' potentials. They are caused by all the charged particles.

$$\partial_i F^{il} = \frac{4\pi}{c} j^l = 4\pi\rho \tag{5}$$

For $l = \alpha$ we have have $4\pi\rho\mathbf{v}/c$, but we won't do this today (tomorrow).

To leading order in v/c, particles only created Coulomb fields and they only "feel" Coulomb fields. Hence to $O((v/c)^0)$, we have

$$\mathcal{L} = \sum_a \frac{m_a \mathbf{v}_a^2}{2} - q_a \phi(\mathbf{x}_a, t) \tag{6}$$

What's the $\phi(\mathbf{x}_a, t)$, the Coulomb field created by all the particles.

How to find?

$$\partial_i F^{i0} = \frac{4\pi}{c} = 4\pi\rho \tag{7}$$

or

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 4\pi\rho = -\boldsymbol{\nabla}^2 \phi \tag{8}$$

where

$$\rho(\mathbf{x}, t) = \sum_a q_a \delta^3(\mathbf{x} - \mathbf{x}_a(t)) \tag{9}$$

This is a Poisson equation

$$\Delta\phi(\mathbf{x}) = \sum_a q_a 4\pi \delta^3(\mathbf{x} - \mathbf{x}_a) \tag{10}$$

(where the time dependence has been suppressed). This has solution

$$\phi(\mathbf{x}, t) = \sum_b \frac{q_b}{|\mathbf{x} - \mathbf{x}_b(t)|} \tag{11}$$

This is the sum of instantaneous Coulomb potentials of all particles at the point of interest. Hence, it appears that $\phi(\mathbf{x}_a, t)$ should be evaluated in 11 at $\mathbf{x}_a$?

However 11 becomes infinite due to contributions of the a-th particle itself. Solution to this is to drop the term, but let's discuss this first.

Let's talk about the electrostatic energy of our system of particles.

$$
\begin{aligned}
\mathcal{E} &= \frac{1}{8\pi}\int d^3\mathbf{x}\left(\mathbf{E}^2 + \cancel{\mathbf{B}^2}\right) \\
&= \frac{1}{8\pi}\int d^3\mathbf{x}\mathbf{E}\cdot(-\boldsymbol{\nabla}\phi) \\
&= \frac{1}{8\pi}\int d^3\mathbf{x}\left(\boldsymbol{\nabla}\cdot(\mathbf{E}\phi) - \phi\boldsymbol{\nabla}\cdot\mathbf{E}\right) \\
&= -\frac{1}{8\pi}\oint d^2\boldsymbol{\sigma}\cdot\mathbf{E}\phi + \frac{1}{8\pi}\int d^3\mathbf{x}\phi\boldsymbol{\nabla}\cdot\mathbf{E}
\end{aligned}
$$

The first term is zero since $\mathbf{E}\phi$ for a localized system of charges $\sim 1/r^3$ or higher as $V \to \infty$. In the second term

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi\sum_a q_a\delta^3(\mathbf{x} - \mathbf{x}_a(t)) \tag{12}$$

So we have

$$\sum_a \frac{1}{2}\int d^3\mathbf{x} q_a\delta^3(\mathbf{x} - \mathbf{x}_a)\phi(\mathbf{x}) \tag{13}$$

for

$$\mathcal{E} = \frac{1}{2}\sum_a q_a\phi(\mathbf{x}_a) \tag{14}$$

Now substitute 11 into 14 for

$$\mathcal{E} = \frac{1}{2}\sum_a \frac{q_a^2}{|\mathbf{x} - \mathbf{x}_a|} + \frac{1}{2}\sum_{a\neq b}\frac{q_a q_b}{|\mathbf{x}_a - \mathbf{x}_b|} \tag{15}$$

or

$$\mathcal{E} = \frac{1}{2}\sum_a \frac{q_a^2}{|\mathbf{x} - \mathbf{x}_a|} + \sum_{a<b}\frac{q_a q_b}{|\mathbf{x}_a - \mathbf{x}_b|} \tag{16}$$

The first term is the sum of the electrostatic self energies of all particles. The source of this infinite self energy is in assuming a point like nature of the particle. i.e. We modeled the charge using a delta function instead of using a continuous charge distribution.

Recall that if you have a charged sphere of radius r

PICTURE: total charge q, radius r, our electrostatic energy is

$$\mathcal{E} \sim \frac{q^2}{r} \tag{17}$$

Stipulate that rest energy $m_e c^2$ is all of electrostatic origin $\sim e^2/r_e$ we get that

$$r_e \sim \frac{e^2}{m_e c^2} \tag{18}$$

This is called the classical radius of the electron, and is of a very small scale 10^{-13}cm.

As a matter of fact the applicability of classical electrodynamics breaks down much sooner than this scale since quantum effects start kicking in.

Our Lagrangian is now

$$\mathcal{L}_a = \frac{1}{2} m_a \mathbf{v}_a^2 - q_a \phi(\mathbf{x}_a, t) \tag{19}$$

where ϕ is the electrostatic potential due to all <u>other</u> particles, so we have

$$\mathcal{L}_a = \frac{1}{2} m_a \mathbf{v}_a^2 - \sum_{a \neq b} \frac{q_a q_b}{|\mathbf{x}_a - \mathbf{x}_b|} \tag{20}$$

and for the system

$$\mathcal{L} = \frac{1}{2} \sum_a m_a \mathbf{v}_a^2 - \sum_{a<b} \frac{q_a q_b}{|\mathbf{x}_a - \mathbf{x}_b|} \tag{21}$$

This is THE Lagrangian for electrodynamics in the non-relativistic case, starting with the relativistic action.

4. What's next?

We continue to the next order of v/c tomorrow.

References

[1] L.D. Landau and E.M. Lifshitz. *The classical theory of fields*. Butterworth-Heinemann, 1980. 1