PHY450H1S. Relativistic Electrodynamics Tutorial 8 (TA: Simon Freedman). EM fields from magnetic dipole current.

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1. Review.

Recall for the electric dipole we started with a system like

$$z_+ = 0 \tag{1}$$
$$z_- = \mathbf{e}_3(z_0 + a\sin(\omega t)) \tag{2}$$

(we did it with the opposite polarity)

$$\mathbf{E} = \frac{qa\omega^2}{c^2}\sin\omega t_o \sin\theta \frac{1}{|\mathbf{x}|}(-\hat{\boldsymbol{\theta}}) = \frac{1}{c^2|\mathbf{x}|}(\ddot{\mathbf{d}}(t_r)\times\hat{\mathbf{r}})\times\hat{\mathbf{r}} \tag{3}$$
$$\mathbf{B} = -\frac{qa\omega^2}{c^2}\sin\omega t_o \sin\theta \frac{1}{|\mathbf{x}|}(-\hat{\boldsymbol{\phi}}) = \hat{\mathbf{r}}\times\mathbf{E}. \tag{4}$$

This was after the multipole expansion ($\lambda \gg l$).

Physical analogy: a high and low frequency wave interacting. The low frequency wave becomes the envelope, and doesn't really "see" the dynamics of the high frequency wave.

We also figured out the Poynting vector was

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} = \hat{\mathbf{r}}\frac{\sin^2\theta\left|\ddot{\mathbf{d}}(t_r)\right|^2}{4\pi c^3|\mathbf{x}|^2}, \tag{5}$$

and our Power was

$$\text{Power}(R) = \oint_{{S_R}^2} d^2\boldsymbol{\sigma}\cdot\langle\mathbf{S}\rangle = \frac{q^2a^2\omega^4}{3c^3}. \tag{6}$$

2. **Magnetic dipole**

PICTURE: positively oriented current I circulating around the normal $\mathbf{m}$ at radius b in the x-y plane. We have
(from third year)

$$|\mathbf{m}| = I\pi b^2. \tag{7}$$

With the magnetic moment directed upwards along the z-axis

$$\mathbf{m} = I\pi b^2 \mathbf{e}_3, \tag{8}$$

where we have a frequency dependence in the current

$$I = I_o \sin(\omega t). \tag{9}$$

With no static charge distribution we have zero scalar potential

$$\rho = 0 \implies A^0 = 0. \tag{10}$$

Our first moments approximation of the vector potential was

$$A^\alpha(\mathbf{x}, t) \approx \frac{1}{c|\mathbf{x}|} \int d^3\mathbf{x}' j^\alpha(\mathbf{x}', t) + O(\text{higher moments}). \tag{11}$$

Now we use our new trick introducing a $1 = 1$ to rewrite the current

$$\left(\frac{\partial x'^\alpha}{\partial x'^\beta}\right) j^\beta = \delta^\alpha{}_\beta j^\beta = j^\alpha, \tag{12}$$

or equivalently

$$\boldsymbol{\nabla} x^\alpha = \mathbf{e}_\alpha. \tag{13}$$

Carrying out the trickery we have

$$\begin{aligned}
A^\alpha &= \frac{1}{c|\mathbf{x}|} \int d^3\mathbf{x}' (\boldsymbol{\nabla}' x'^\alpha) \cdot \mathbf{J}(\mathbf{x}', t_r) \\
&= \frac{1}{c|\mathbf{x}|} \int d^3\mathbf{x}' (\partial_{\beta'} x'^\alpha) j^\beta(\mathbf{x}', t_r) \\
&= \frac{1}{c|\mathbf{x}|} \int d^3\mathbf{x}' (\partial_{\beta'} (x'^\alpha j^\beta(\mathbf{x}', t_r)) - x'^\alpha (\underbrace{\boldsymbol{\nabla}' \cdot \mathbf{J}(\mathbf{x}', t_r)}_{=-\partial_0 \rho = 0})) \\
&= \frac{1}{c|\mathbf{x}|} \int d^3\mathbf{x}' \boldsymbol{\nabla}' \cdot (x'^\alpha \mathbf{J}) \\
&= \oint_{S_R{}^2} d^2\boldsymbol{\sigma} \cdot (x'^\alpha \mathbf{J}) \\
&= 0.
\end{aligned}$$

We see that the first order approximation is insufficient to calculate the vector potential for the magnetic dipole system, and that we have

$$A^{\alpha} = 0 + \text{higher moments} \tag{14}$$

Looking back to what we'd done in class, we'd also dropped this term of the vector potential, using the same arguments. What we had left was

$$\mathbf{A}(\mathbf{x}, t) = \frac{1}{c|\mathbf{x}|}\dot{\mathbf{d}}\left(t - \frac{|\mathbf{x}|}{c}\right) = \frac{1}{c|\mathbf{x}|}\int d^3\mathbf{x}' x'^{\alpha} \frac{\partial}{\partial t}\rho\left(\mathbf{x}', t - \frac{|\mathbf{x}|}{c}\right), \tag{15}$$

but that additional term is also zero in this magnetic dipole system since we have no static charge distribution.

There are two options to resolve this

1. calculate $\mathbf{A}$ using higher order moments $\lambda \gg b$. Go to next order in b/λ.

   This is complicated!

2. Use EM dualities (the slick way!)

Recall that Maxwell's equations are

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 4\pi\rho \tag{16}$$
$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{17}$$
$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \tag{18}$$
$$\boldsymbol{\nabla} \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + 4\pi\mathbf{J} \tag{19}$$

If $j^i = 0$, then taking $\mathbf{E} \rightarrow \mathbf{B}$ and $\mathbf{B} \rightarrow \mathbf{E}$ we get the same equations.
Introduce dual charges ρ_m and $\mathbf{J}_m$

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 4\pi\rho_e \tag{20}$$
$$\boldsymbol{\nabla} \cdot \mathbf{B} = 4\pi\rho_m \tag{21}$$
$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} + 4\pi\mathbf{J}_m \tag{22}$$
$$\boldsymbol{\nabla} \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + 4\pi\mathbf{J}_e \tag{23}$$

Duality $\mathbf{E} \rightarrow \mathbf{B}$ provided $\rho_e \rightarrow \rho_m$ and $\mathbf{J}_e \rightarrow \mathbf{J}_m$, or

$$F^{ij} \rightarrow \tilde{F}^{ij} = \epsilon^{ijkl}F_{kl} \tag{24}$$
$$j^k \rightarrow \tilde{j}^k \tag{25}$$

With radiation : the duality transformation takes the electric dipole moment to the magnetic dipole moment $\mathbf{d} \rightarrow \mathbf{m}$.

$$\mathbf{B} = -\frac{1}{c^2|\mathbf{x}|}(\ddot{\mathbf{m}} \times \hat{\mathbf{r}}) \times \hat{\mathbf{r}} \tag{26}$$

$$\mathbf{E} = \hat{\mathbf{r}} \times \mathbf{B} \tag{27}$$

with

$$\text{Power} \sim \left\langle \left| \ddot{\mathbf{m}}^2 \right| \right\rangle \tag{28}$$

$$\left\langle \left| \ddot{\mathbf{m}}^2 \right| \right\rangle = \frac{1}{2}(I_o \pi b^2 \omega^2)^2 \tag{29}$$

where

$$I_o = \dot{q} = \omega q \tag{30}$$

So the power of the magnetic dipole is

$$P_m(R) = \frac{b^4 q^2 \pi^2 \omega^6}{3c^5} \tag{31}$$

Taking ratios of the magnetic and electric power we find

$$\begin{aligned}
\frac{P_m}{E_m} &= \frac{b^4 q^2 \pi^2 \omega^6}{b^2 q^2 \omega^4 c^2} \\
&\sim \frac{b^2 \omega^2}{c^2} \\
&= \left(\frac{b\omega}{c}\right)^2 \\
&= \left(\frac{b}{\lambda}\right)^2
\end{aligned}$$

This difference in power shows the second order moment dependence, in the $\lambda \gg b$ approximations.

FIXME: go back and review the "third year" content and see where the magnetic dipole moment came from. That's the key to this argument, since we need to see how this ends up equivalent to a pair of charges in the electric field case.

3. **Midterm solution discussion.**

In the last part of the tutorial, the bonus question from the tutorial was covered. This was to determine the Yukawa potential from the differential equation that we found in the earlier part of the problem.

I took a couple notes about this on paper, but don't intend to write them up. Everything proceeded exactly as I would have expected them to for solving the problem (I barely finished the midterm as is, so I didn't have a chance to try it). Take Fourier transforms and then evaluate the inverse Fourier integral. This is exactly what we can do for the Coulomb potential, but actually easier since we don't have to introduce anything to offset the poles (and we recover the Coulomb potential in the $M \to 0$ case).