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Evaluating the squared sinc integral

1.1 Motivation

In the Fermi's golden rule lecture we used the result for the integral of the squared sinc function. Here is a reminder of the contours required to perform this integral.

1.2 Guts

We want to evaluate

$$\int_{-\infty}^{\infty} \frac{\sin^2(x|\mu|)}{x^2} dx \tag{1.1}$$

We make a few change of variables

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\sin^2(x|\mu|)}{x^2} dx &= |\mu| \int_{-\infty}^{\infty} \frac{\sin^2(y)}{y^2} dy \\
&= -i|\mu| \int_{-\infty}^{\infty} \frac{(e^{iy} - e^{-iy})^2}{(2iy)^2} i dy \\
&= -\frac{i|\mu|}{4} \int_{-i\infty}^{i\infty} \frac{e^{2z} + e^{-2z} - 2}{z^2} dz
\end{aligned} \tag{1.2}$$

Now we pick a contour that is distorted to one side of the origin as in fig. 1.1

We employ Jordan's theorem (§8.12 [1]) now to pick the contours for each of the integrals since we need to ensure the $e^{\pm z}$ terms converges as $R \rightarrow \infty$ for the $z = Re^{i\theta}$ part of the contour. We can write

$$\int_{-\infty}^{\infty} \frac{\sin^2(x|\mu|)}{x^2} dx = -\frac{i|\mu|}{4} \left(\int_{C_0+C_2} \frac{e^{2z}}{z^2} dz + \int_{C_0+C_1} \frac{e^{-2z}}{z^2} dz - \int_{C_0+C_1} \frac{2}{z^2} dz \right) \tag{1.3}$$

The second two integrals both surround no poles, so we have only the first to deal with

$$\begin{aligned}
\int_{C_0+C_2} \frac{e^{2z}}{z^2} dz &= 2\pi i \frac{1}{1!} \frac{d}{dz} e^{2z} \Big|_{z=0} \\
&= 4\pi i
\end{aligned} \tag{1.4}$$

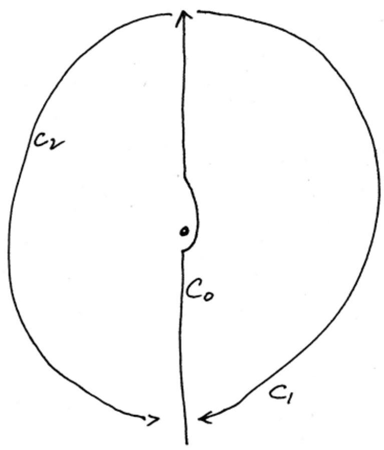

Figure 1.1: Contour distorted to one side of the double pole at the origin

Putting everything back together we have

$$\int_{-\infty}^{\infty} \frac{\sin^2(x|\mu|)}{x^2} dx = -\frac{i|\mu|}{4} 4\pi i = \pi|\mu| \tag{1.5}$$

1.2.1 *On the cavalier choice of contours*

The choice of which contours to pick above may seem pretty arbitrary, but they are for good reason. Suppose you picked $C_0 + C_1$ for the first integral. On the big C_1 arc, then with a $z = Re^{i\theta}$ substitution we have

$$\begin{aligned}
\left| \int_{C_1} \frac{e^{2z}}{z^2} dz \right| &= \left| \int_{\theta=\pi/2}^{-\pi/2} \frac{e^{2R(\cos\theta + i\sin\theta)}}{R^2 e^{2i\theta}} Rie^{i\theta} d\theta \right| \\
&= \frac{1}{R} \left| \int_{\theta=\pi/2}^{-\pi/2} e^{2R(\cos\theta + i\sin\theta)} e^{-i\theta} d\theta \right| \\
&\le \frac{1}{R} \int_{\theta=-\pi/2}^{\pi/2} \left| e^{2R\cos\theta} \right| d\theta \\
&\le \frac{\pi e^{2R}}{R}
\end{aligned} \tag{1.6}$$

This clearly doesn't have the zero convergence property that we desire. We need to pick the C_2 contour for the first (positive exponent) integral since in that $[\pi/2, 3\pi/2]$ range, $\cos\theta$ is always negative. We can however, use the C_1 contour for the second (negative exponent) integral. Explicitly, again by example, using C_2 contour for the first integral, over that portion of the arc we have

$$
\begin{aligned}
\left|\int_{C_2} \frac{e^{2z}}{z^2} dz\right| &= \left|\int_{\theta=\pi/2}^{3\pi/2} \frac{e^{2R(\cos\theta + i\sin\theta)}}{R^2 e^{2i\theta}} Rie^{i\theta} d\theta\right| \\
&= \frac{1}{R}\left|\int_{\theta=\pi/2}^{3\pi/2} e^{2R(\cos\theta + i\sin\theta)} e^{-i\theta} d\theta\right| \\
&\leq \frac{1}{R}\int_{\theta=\pi/2}^{3\pi/2} \left|e^{2R\cos\theta} d\theta\right| \\
&\approx \frac{1}{R}\int_{\theta=\pi/2}^{3\pi/2} \left|e^{-2R} d\theta\right| \\
&= \frac{\pi e^{-2R}}{R}
\end{aligned}
\tag{1.7}
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Bibliography

[1] W.R. Le Page and W.R. LePage. *Complex Variables and the Laplace Transform for Engineers*. Courier Dover Publications, 1980. 1.2