On tensor product generators of the gamma matrices.

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1. Motivation.

In [1] he writes

$$\begin{aligned}
\gamma^0 &= \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} = I \otimes \tau_3 \\
\gamma^i &= \begin{bmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{bmatrix} = \sigma^i \otimes i\tau_2 \\
\gamma^5 &= \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = I \otimes \tau_1
\end{aligned}$$

The Pauli matrices σ^i I had seen, but not the τ_i matrices, nor the $\otimes$ notation. Strangerep in physicsforums points out that the $\otimes$ is a Kronecker matrix product, a special kind of tensor product [2]. Let's do the exersize of reverse engineering the τ matrices as suggested.

2. Guts

Let's start with γ^5. We want

$$\gamma^5 = I \otimes \tau_1 = \begin{bmatrix} I\tau_{11} & I\tau_{12} \\ I\tau_{21} & I\tau_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{1}$$

By inspection we must have

$$\tau_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \sigma^1 \tag{2}$$

Thus $\tau_1 = \sigma^1$. How about τ_2? For that matrix we have

$$\gamma^i = \sigma^i \otimes i\tau_2 = \begin{bmatrix} \sigma^i\tau_{11} & \sigma^i\tau_{12} \\ \sigma^i\tau_{21} & \sigma^i\tau_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{3}$$

Again by inspection we must have

$$i\tau_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{4}$$

so

$$\tau_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \sigma^2. \tag{5}$$

This one is also just the Pauli matrix. For the last we have

$$\gamma^0 = I \otimes \tau_3 = \begin{bmatrix} I\tau_{11} & I\tau_{12} \\ I\tau_{21} & I\tau_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{6}$$

Our last tau matrix is thus

$$\tau_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sigma^3. \tag{7}$$

Curious that there are two notations used in the same page for exactly the same thing? It appears that I wasn't the only person confused about this.

3. The bivector expansion

Zee writes his wedge products with the commutator, adding a complex factor

$$\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right] \tag{8}$$

Let's try the direct product notation to expand σ^{0i} and σ^{ij}. That first is

$$\begin{aligned}
\sigma^{0i} &= \frac{i}{2}\left(\gamma^0\gamma^i - \gamma^i\gamma^0\right) \\
&= i\gamma^0\gamma^i \\
&= i(I \otimes \tau_3)(\sigma^i \otimes i\tau_2) \\
&= i^2\sigma^i \otimes \tau_3\tau_2 \\
&= -\sigma^i \otimes (-i\tau_1) \\
&= i\sigma^i \otimes \tau_1 \\
&= i \begin{bmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{bmatrix},
\end{aligned}$$

which is what was expected. The second bivector, for $i = j$ is zero, and for $i \neq j$ is

$$\begin{aligned}
\sigma^{ij} &= i\gamma^i\gamma^j \\
&= i(\sigma^i \otimes i\tau_2)(\sigma^j \otimes i\tau_2) \\
&= i^3(\sigma^i\sigma^j) \otimes I \\
&= i^4(\epsilon_{ijk}\sigma^k) \otimes I \\
&= \epsilon_{ijk} \begin{bmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{bmatrix}.
\end{aligned}$$

References

[1] A. Zee. *Quantum field theory in a nutshell*. Universities Press, 2005. 1

[2] Wikipedia. Tensor product — wikipedia, the free encyclopedia [online]. 2011. [Online; accessed 21-June-2011]. Available from: http://en.wikipedia.org/w/index.php?title=Tensor_product&oldid=418002023. 1