PHY454H1S
Continuum Mechanics. Lecture 8: Phasor description of elastic waves. Fluid dynamics. Taught by Prof. K. Das.

Originally appeared at:
`http://sites.google.com/site/peeterjoot2/math2012/continuumL8.pdf`

Peeter Joot — peeter.joot@gmail.com
Feb 3, 2012 continuumL8.tex

Contents

1 Disclaimer. **1**

2 Review. Elastic wave equation **1**
2.1 Some waves illustrated. 3

3 Fluid dynamics. **3**
3.1 Making sense of things. 5

1. Disclaimer.

Peeter's lecture notes from class. May not be entirely coherent.

2. Review. Elastic wave equation

Starting with

$$\rho \frac{\partial^2 \mathbf{e}}{\partial t^2} = (\lambda + \mu) \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \mathbf{e}) + \mu \boldsymbol{\nabla}^2 \mathbf{e} \tag{1}$$

and applying a divergence operation we find

$$\rho \frac{\partial^2 \theta}{\partial t^2} = C_L^2 \boldsymbol{\nabla}^2 \theta \tag{2}$$

$$\theta = \boldsymbol{\nabla} \cdot \mathbf{e} \tag{3}$$

$$C_L^2 = \frac{\lambda + 2\mu}{\rho}. \tag{4}$$

This is the P-wave equation. Applying a curl operation we find

$$\rho \frac{\partial^2 \boldsymbol{\omega}}{\partial t^2} = C_T^2 \boldsymbol{\nabla}^2 \boldsymbol{\omega} \tag{5}$$

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{e} \tag{6}$$

$$C_T^2 = \frac{\lambda + 2\mu}{\rho}. \tag{7}$$

This is the S-wave equation. We also found that

$$\frac{C_L}{C_T} > 1, \tag{8}$$

and concluded that P waves are faster than S waves. What we haven't shown is that the P waves are longitudinal, and that the S waves are transverse.

Assuming a gradient and curl description of our displacement

$$\mathbf{e} = \boldsymbol{\nabla}\phi + \boldsymbol{\nabla} \times \mathbf{H} = \mathbf{P} + \mathbf{S}, \tag{9}$$

we found

$$(\lambda + 2\mu)\boldsymbol{\nabla}^2\phi - \rho\frac{\partial^2\phi}{\partial t^2} = 0 \tag{10}$$

$$\mu\boldsymbol{\nabla}^2\mathbf{H} - \rho\frac{\partial^2\mathbf{H}}{\partial t^2} = 0, \tag{11}$$

allowing us to separately solve for the P and the S wave solutions respectively. Now, let's introduce a phasor representation (again following §22 of the text [1])

$$\phi = A\exp\left(i(\mathbf{k}\cdot\mathbf{x} - \omega t)\right) \tag{12}$$

$$\mathbf{H} = \mathbf{B}\exp\left(i(\mathbf{k}\cdot\mathbf{x} - \omega t)\right) \tag{13}$$

Operating with the gradient we find

$$\begin{aligned}
\mathbf{P} &= \boldsymbol{\nabla}\phi \\
&= \mathbf{e}_k\partial_k A\exp\left(i(\mathbf{k}\cdot\mathbf{x} - \omega t)\right) \\
&= \mathbf{e}_k\partial_k A\exp\left(i(k_m x_m - \omega t)\right) \\
&= \mathbf{e}_k ik_k A\exp\left(i(k_m x_m - \omega t)\right) \\
&= i\mathbf{k}A\exp\left(i(\mathbf{k}\cdot\mathbf{x} - \omega t)\right) \\
&= i\mathbf{k}\phi
\end{aligned}$$

We can also write

$$\mathbf{P} = \mathbf{k}\phi' \tag{14}$$

where ϕ' is the derivative of ϕ "with respect to its argument". Here argument must mean the entire phase $\mathbf{k}\cdot\mathbf{x} - \omega t$.

$$\phi' = \frac{d\phi(\mathbf{k}\cdot\mathbf{x} - \omega t)}{d(\mathbf{k}\cdot\mathbf{x} - \omega t)} = i\phi \tag{15}$$

Actually, argument is a good label here, since we can use the word in the complex number sense.

For the curl term we find

$$
\begin{aligned}
\mathbf{S} &= \boldsymbol{\nabla} \times \mathbf{H} \\
&= \mathbf{e}_a \partial_b H_c \epsilon_{abc} \\
&= \mathbf{e}_a \partial_b \epsilon_{abc} B_c \exp\left(i(\mathbf{k} \cdot \mathbf{x} - \omega t)\right) \\
&= \mathbf{e}_a \partial_b \epsilon_{abc} B_c \exp\left(i(k_m x_m - \omega t)\right) \\
&= \mathbf{e}_a i k_b \epsilon_{abc} B_c \exp\left(i(\mathbf{k} \cdot \mathbf{x} - \omega t)\right) \\
&= i\mathbf{k} \times \mathbf{H}
\end{aligned}
$$

Again writing

$$\mathbf{H}' = \frac{d\mathbf{H}(\mathbf{k} \cdot \mathbf{x} - \omega t)}{d(\mathbf{k} \cdot \mathbf{x} - \omega t)} = i\mathbf{H} \tag{16}$$

we can write the S wave as

$$\mathbf{S} = \mathbf{k} \times \mathbf{H}' \tag{17}$$

2.1. *Some waves illustrated.*

The following wave types were noted, but not defined:

- Rayleigh wave. This is discussed in §24 of the text (a wave that propagates near the surface of a body without penetrating into it). Wikipedia has an illustration of one possible mode of propagation [2].
- Love wave. These aren't discussed in the text, but wikipedia [3] describes them as polarized shear waves (where the figure indicates that the shear displacements are perpendicular to the direction of propagation).

Some illustrations from the class notes were also shown. Hopefully we'll have some homework assignments where we do some problems to get a feel for how to apply the formalism.

3. **Fluid dynamics.**

In fluid dynamics we look at displacements with respect to time as illustrated in figure (1)

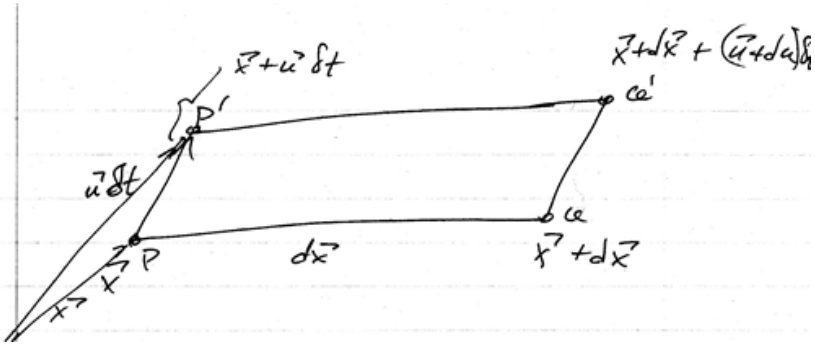

Figure 1: Differential displacement.

$$d\mathbf{x}' = d\mathbf{x} + d\mathbf{u}\delta t \tag{18}$$

In index notation

$$\begin{aligned} dx_i' &= dx_i + du_i\delta t \\ &= dx_i + \frac{\partial u_i}{\partial x_j} dx_j \delta t \end{aligned}$$

We define

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{19}$$

a symmetric tensor. We also define

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \tag{20}$$

Effect of e_{ij} when diagonalized

$$e_{ij} == \begin{bmatrix} e_{11} & 0 & 0 \\ 0 & e_{22} & 0 \\ 0 & 0 & e_{33} \end{bmatrix} \tag{21}$$

so that in this frame of reference we have

$$dx_1' = (1 + e_{11}\delta t)dx_1 \tag{22}$$
$$dx_2' = (1 + e_{22}\delta t)dx_2 \tag{23}$$
$$dx_3' = (1 + e_{33}\delta t)dx_3 \tag{24}$$

Let's find the matrix form of the antisymmetric tensor. We find

$$\omega_{11} = \omega_{22} = \omega_{33} = 0 \tag{25}$$

Introducing a vorticity vector

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u} \tag{26}$$

we find

$$\omega_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) = -\frac{1}{2}(\boldsymbol{\nabla} \times \mathbf{u})_3 \tag{27}$$
$$\omega_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) = -\frac{1}{2}(\boldsymbol{\nabla} \times \mathbf{u})_1 \tag{28}$$
$$\omega_{31} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right) = -\frac{1}{2}(\boldsymbol{\nabla} \times \mathbf{u})_2 \tag{29}$$

Writing

$$\Omega_i = \frac{1}{2}\omega_i \tag{30}$$

we find the matrix form of this antisymmetric tensor

$$\omega_{ij} = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix} \tag{31}$$

$$\begin{aligned} dx_1' &= dx_1 + (\cancel{\omega_{11}} dx_1 + \omega_{12} dx_2 + \omega_{13} dx_3)\,\delta t \\ &= dx_1 + (\omega_{12} dx_2 + \omega_{13} dx_3)\,\delta t \\ &= dx_1 + (\Omega_2 dx_3 - \Omega_3 dx_2)\,\delta t \end{aligned}$$

Doing this for all components we find

$$d\mathbf{x}' = d\mathbf{x} + (\mathbf{\Omega} \times d\mathbf{x})\delta t. \tag{32}$$

The tensor ω_{ij} implies rotation of a control volume with an angular velocity $\mathbf{\Omega} = \boldsymbol{\omega}/2$ (half the vorticity vector).

In general we have

$$dx_i' = dx_i + e_{ij} dx_j \delta t + \omega_{ij} dx_j \delta t \tag{33}$$

*3.1. **Making sense of things.***

After this first fluid dynamics lecture I was left troubled. We'd just been barraged with a set of equations pulled out of a magic hat, with no notion of where they came from. Unlike the contiuum strain tensor, which was derived by considering differences in squared displacements, we have an antisymmetric term now. Why did we have no such term considering solids?

After a bit of thought I think I see where things are coming from. We have essentially looked at a first order decomposition of the displacement (per unit time) of a point in terms of symmetric and antisymmetric terms. This is really just a gradient evaluation, split into coordinates

$$\begin{aligned} x_i' &= x_i + (\boldsymbol{\nabla} u_i) \cdot d\mathbf{x}\delta t \\ &= x_i + \frac{\partial u_i}{\partial x_j} dx_j \delta t \\ &= x_i + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j}\right) dx_j \delta t + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_i}{\partial x_j}\right) dx_j \delta t \\ &= x_i + e_{ij} dx_j \delta t + \omega_{ij} dx_j \delta t \end{aligned}$$

Here, as in the solids case, we have

$$\mathbf{u} = \mathbf{x}' - \mathbf{x} \tag{34}$$

References

[1] L.D. Landau, EM Lifshitz, JB Sykes, WH Reid, and E.H. Dill. Theory of elasticity: Vol. 7 of course of theoretical physics. 1960. 2

[2] Wikipedia. Rayleigh wave — wikipedia, the free encyclopedia [online]. 2012. [Online; accessed 4-February-2012]. Available from: http://en.wikipedia.org/w/index.php?title=Rayleigh_wave&oldid=473693354. 2.1

[3] Wikipedia. Love wave — wikipedia, the free encyclopedia [online]. 2012. [Online; accessed 4-February-2012]. Available from: http://en.wikipedia.org/w/index.php?title=Love_wave&oldid=474355253. 2.1