PHY454H1S
Continuum Mechanics. Lecture 9: Newtonian fluids. Mass conservation. Constitutive relation. Incompressible fluids. Taught by Prof. K. Das.

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Contents

1. Reading

§1.4 from [1]. FIXME: Probably more elsewhere too.

2. Disclaimer.

Peeter's lecture notes from class. May not be entirely coherent.

3. Review: Relative motion near a point in a fluid

Referring to figure (1)
we write

$$d\mathbf{x}' = d\mathbf{x} + d\mathbf{u}\delta t \tag{1}$$

or in coordinate form

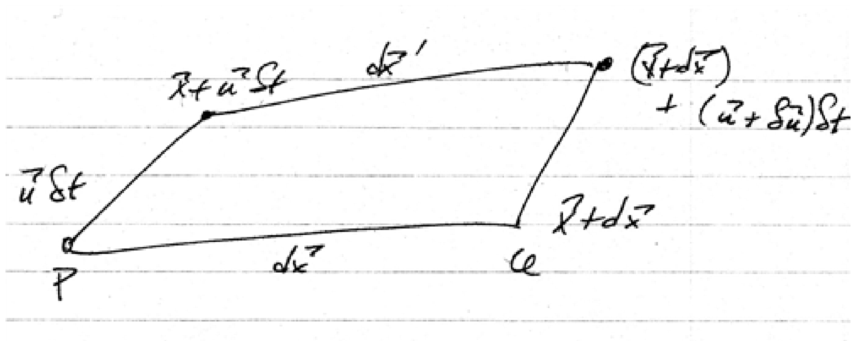

Figure 1: velocity displacements at a fluid point.

$$
\begin{aligned}
dx_i &= dx_i + du_i \delta t \\
&= dx_i + \frac{\partial u_i}{\partial x_j} dx_j \delta t
\end{aligned} \tag{2}
$$

We can now split the components of the gradient of u_i into symmetric and antisymmetric parts in the normal way

$$
\begin{aligned}
\frac{\partial u_i}{\partial x_j} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \\
&\equiv e_{ij} + \omega_{ij}.
\end{aligned} \tag{3}
$$

3.1. *The antisymmetric term (name?)*

With

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u}, \tag{4}$$

we introduce the dual vector

$$\boldsymbol{\Omega} = \Omega_k \mathbf{e}_k = \frac{1}{2}\boldsymbol{\omega} \tag{5}$$

defined according to

$$\Omega_1 = \frac{1}{2}\omega_{32} = \frac{1}{2}\omega_1 \tag{6}$$

$$\Omega_2 = \frac{1}{2}\omega_{13} = \frac{1}{2}\omega_2 \tag{7}$$

$$\Omega_3 = \frac{1}{2}\omega_{21} = \frac{1}{2}\omega_3 \tag{8}$$

With

$$\omega_{ij} = \epsilon_{ijk}\partial_i u_j \tag{9}$$

we can write

$$\Omega_k = -\frac{1}{2}\epsilon_{ijk}\partial_i u_j. \tag{10}$$

In matrix form this becomes

$$\omega_{ij} = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix}. \tag{11}$$

For the special case $e_{ij} = 0$, our displacement equation in vector form becomes

$$d\mathbf{x}' = d\mathbf{x} + \mathbf{\Omega} \times d\mathbf{x}\delta t. \tag{12}$$

Let's do a quick verification that this is all kosher.

$$\begin{aligned}
(\mathbf{\Omega} \times d\mathbf{x})_i &= \Omega_r dx_s \epsilon_{rsi} \\
&= \left(-\frac{1}{2}\epsilon_{abr}\partial_a u_b\right) dx_s \epsilon_{rsi} \\
&= -\frac{1}{2}\partial_a u_b dx_s \delta_{si}^{[ab]} \\
&= -\frac{1}{2}(\partial_s u_i - \partial_i u_s) dx_s \\
&= \frac{1}{2}\left(\frac{\partial u_s}{\partial x_i} - \frac{\partial u_i}{\partial x_s}\right) dx_s \\
&= \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right) dx_j \\
&= \omega_{ij} dx_j.
\end{aligned}$$

All's good in the world of signs and indexes.

3.2. *The symmetric term (strain tensor).*

Now let's look at the symmetric term. With the initial volume

$$dV = dx_1 dx_2 dx_3, \tag{13}$$

and the final volume written assuming that we are working in our principle strain basis, we have (very much like the solids case)

$$\begin{aligned}
dV' &= dx_1' dx_2' dx_3' \\
&= (1 + e_{11}\delta t)dx_1 + (1 + e_{22}\delta t)dx_2 + (1 + e_{33}\delta t)dx_3 \\
&= (1 + (e_{11} + e_{22} + e_{33})\delta t)dx_1 dx_2 dx_3 + O((\delta t)^2) \\
&= \left(1 + \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)\delta t\right) dV \\
&= (1 + (\boldsymbol{\nabla} \cdot \mathbf{u})\delta t)\, dV
\end{aligned}$$

So much like we expressed the relative change of volume in solids, we now can express the relative change of volume per unit time as

$$\frac{dV' - dV}{dV\delta t} = \boldsymbol{\nabla} \cdot \mathbf{u}, \tag{14}$$

or

$$\frac{\delta(dV)}{dV\delta t} = \boldsymbol{\nabla} \cdot \mathbf{u}, \tag{15}$$

We identify the divergence of the displacement as the relative change in volume per unit time.

4. Newtonian Fluids.

Definition 4.1 (Newtonian Fluids) *A fluid for which the rate of strain tensor is linearly related to stress tensor.*

For such a fluid, the constitutive relation takes the form

$$\boxed{\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij},} \tag{16}$$

where p is called the isotropic pressure, and μ is the viscosity of the fluid.
For comparison, in solids we had

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} \tag{17}$$

While we are allowing for rotation in the fluids (ω_{ij}) that we did not consider for solids, we now impose a requirement that the strain tensor trace is not a function of the fluid displacements, with

$$\lambda e_{kk} = \lambda \boldsymbol{\nabla} \cdot \mathbf{u} = -p. \tag{18}$$

What is the physical justification for this? In words this was explained after class as the effect of rotation invariance with an attempt to measure the pressure at a given point in the fluid. It doesn't matter what direction we place our pressure measurement device at a given fixed location in the fluid. Note that this doesn't mean the pressure itself is constant. For example with a gravitational body force applied, our pressure will increase with depth in the fluid. Noting this provides a nice physical interpretation of the trace of the strain tensor.
Can we mathematically justify this explanation? We see above that we have

$$\boldsymbol{\nabla} \cdot \mathbf{u} = \frac{\delta \ln(dV)}{\delta t}, \tag{19}$$

so we are in effect making the identification

$$\ln dV = -pt/\lambda + \ln dV_0 \tag{20}$$

or

$$dV = dV_0 e^{-pt/\lambda}. \tag{21}$$

The relative change in a differential volume element changes exponentially.

4.1. *Dimensions*

$$[\mu] = \frac{\mathrm{M}}{\mathrm{LT}}. \tag{22}$$

Some examples

- $\mu_{\mathrm{air}} = 1.8 \times 10^{-5} \frac{\mathrm{kg}}{\mathrm{m\,s}}$
- $\mu_{\mathrm{water}} = 1.1 \times 10^{-3} \frac{\mathrm{kg}}{\mathrm{m\,s}}$
- $\mu_{\mathrm{glycerin}} = 2.3 \frac{\mathrm{kg}}{\mathrm{m\,s}}$

5. **Conservation of mass in fluid.**

Referring to figure (2)

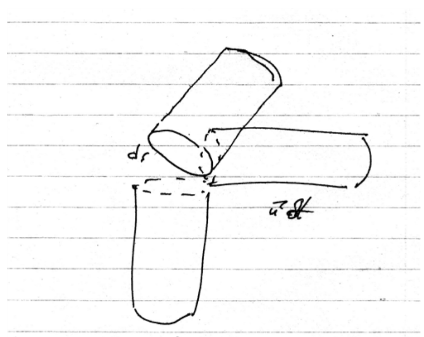

Figure 2: FIXME: continuumL9fig2

we have a flow rate

$$\rho \mathbf{u} \delta t ds \tag{23}$$

or

$$\rho \mathbf{u} ds, \tag{24}$$

per unit time. Here the velocity of fluid particle is $\mathbf{u}$.

$$\oint \rho \mathbf{u} \cdot d\mathbf{s}, \tag{25}$$

we must have

$$\frac{\partial}{\partial t} \int \rho dV = -\oint \rho \mathbf{u} \cdot d\mathbf{s}. \tag{26}$$

$$dm = \rho dV \tag{27}$$

$$\frac{dm}{dt} = \frac{d}{dt}(\rho dV) \tag{28}$$

- positive if fluid is coming in.
- negative if fluid is going out.

By Green's theorem

$$\oint \mathbf{A} \cdot d\mathbf{s} = \int_V (\boldsymbol{\nabla} \cdot \mathbf{u}) dV, \tag{29}$$

so we have

$$-\oint \rho\mathbf{u} \cdot d\mathbf{s} = -\int \boldsymbol{\nabla} \cdot (\rho\mathbf{u}) dV, \tag{30}$$

and must have

$$\int \left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\mathbf{u}) \right) dV = 0. \tag{31}$$

The total mass has to be conserved. The mass that is leaving the volume per unit time must move through the surface of the volume in that time. In differential form this is

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\mathbf{u}) = 0.} \tag{32}$$

Operating by chain rule we can write this as

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla}\rho = -\rho\boldsymbol{\nabla} \cdot \mathbf{u}. \tag{33}$$

To make sense of this, observe that we have for $f = f(x, y, z, t)$

$$\begin{aligned}
\delta f &= \lim_{\delta t \to 0} \frac{\delta f}{\delta t} dt \\
&= \frac{\partial f}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial f}{\partial y}\frac{\delta y}{\delta t} + \frac{\partial f}{\partial z}\frac{\delta z}{\delta t} + \frac{\partial f}{\partial t} \\
&= (\boldsymbol{\nabla} f) \cdot \mathbf{u} + \frac{\partial f}{\partial t}
\end{aligned}$$

so we have

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla}\rho = \frac{d\rho}{dt} \tag{34}$$

or

$$\frac{d\rho}{dt} = -\rho\boldsymbol{\nabla} \cdot \mathbf{u}. \tag{35}$$

5.1. ***Incompressible fluid***

When the density doesn't change note that we have

$$\frac{d\rho}{dt} = 0 \tag{36}$$

which then implies

$$\boxed{\nabla \cdot \mathbf{u} = 0,} \tag{37}$$

at all points in the fluid.

References

[1] D.J. Acheson. *Elementary fluid dynamics*. Oxford University Press, USA, 1990. 1