PHY454H1S Continuum mechanics. Problem Set 3. Velocity scaling, non-dimensionalisation, boundary layers.

`http://sites.google.com/site/peeterjoot2/math2012/continuumProblemSet3.pdf`

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1. Disclaimer.

This problem set is as yet ungraded.

2. Problem Q1.

2.1. *Background.*

In fluid convection problems one can make several choices for characteristic velocity scales. Some choices are given below for example:

1. $U_1 = g\alpha d^2 \nabla T / \nu$
2. $U_2 = \nu / d$
3. $U_3 = \sqrt{g\alpha d \nabla T}$
4. $U_4 = \kappa / d$

where g is the acceleration due to gravity, $\alpha = (\partial V / \partial T)/V$ is the coefficient of volume expansion, d length scale associated with the problem, ∇T is the applied temperature difference, ν is the kinematic viscosity and κ is the thermal diffusivity.

1. For U_1:

   Observing that

$$\left[\frac{\partial \mathbf{u}}{\partial t} \right] = [\nu \boldsymbol{\nabla}^2 \mathbf{u}] \tag{1}$$

   we must have

$$[\nu] = \frac{1}{[t][\boldsymbol{\nabla}^2]} = \frac{1}{T} L^2 \tag{2}$$

   We also find

$$[\alpha] = \frac{1}{[V]} \left[\frac{\partial V}{\partial T} \right] = \frac{1}{[K]}, \tag{3}$$

so that

$$[U_1] = \frac{L}{T\cancel{T}}\frac{\cancel{1}}{\cancel{K}}L^2\cancel{K}\frac{\cancel{T}}{L^2} = \frac{L}{T} \tag{4}$$

2. For U_2:

$$[U_2] = \frac{L^2}{T}\frac{1}{L} = \frac{L}{T}. \tag{5}$$

3. For U_3

$$[U_3] = \sqrt{\frac{L}{T^2}\frac{1}{K}LK} = \frac{L}{T} \tag{6}$$

4. For U_4

   According to http://scienceworld.wolfram.com/physics/ThermalDiffusivity.html, the thermal diffusivity is defined by

$$\frac{\partial T}{\partial t} = \kappa \boldsymbol{\nabla}^2 T \tag{7}$$

   so that

$$[\kappa] = \frac{1}{[t][\boldsymbol{\nabla}^2]} = \frac{L^2}{T} \tag{8}$$

   That gives us

$$[U_4] = \frac{L^2}{T}\frac{1}{L} = \frac{L}{T}. \tag{9}$$

We've verified that all of these have dimensions of velocity.

2.2. *Part 1. Statement. Check whether the dimensions match in each case above.*

2.3. *Solution.*

2.4. *Part 2. Statement. Pure liquid.*

For pure liquid, say pure water at room temperature, one has the following estimates in cgs units:

$$\begin{aligned}\alpha &\sim 10^{-4}\\ \kappa &\sim 10^{-3}\\ \nu &\sim 10^{-2}\end{aligned}$$

For a $d \sim 1$cm layer depth and a ten degree temperature drop convective velocities have been experimentally measured of about 10^{-2}. With $g \sim 10^{-3}$, calculate the values of U_1, U_2, U_3, and U_4.

Which ones of the characteristic velocities (U_1, U_2, U_3, U_4) do you think are suitable for nondimensionalising the velocity in Navier-Stokes/Energy equation describing the water convection problem?

We have

$$U_1 \sim 10^{-3}10^{-4}(1)^2 10^1 \frac{1}{10^{-2}} = 10^{-4} \tag{10}$$

$$U_2 \sim 10^{-2}/1 = 10^{-2} \tag{11}$$

$$U_3 \sim \sqrt{10^{-3}10^{-4}(1)10^1} = 10^{-3} \tag{12}$$

$$U_4 \sim 10^{-3}/1 = 10^{-3} \tag{13}$$

Use of $U_2 = \nu/d$ gives the closest match to the measured characteristic velocity of 10^{-2}.

2.5. Solution.

2.6. Part 3. Statement. Mantle convection.

For mantle convection, we have

$$\begin{aligned} \alpha &\sim 10^{-5} \\ \nu &\sim 10^{21} \\ \kappa &\sim 10^{-2} \\ d &\sim 10^{8} \\ \nabla T &\sim 10^{3}, \end{aligned}$$

and the actual convective mantle velocity is 10^{-8}. Which of the characteristic velocities should we use to nondimensionalise Navier-Stokes/Energy equations describing mantle convection?

2.7. Solution

Let's compute the characteristic velocities again with the mantle numbers

$$U_1 \sim \frac{10^{-3}10^{-5}10^{16}10^{3}}{10^{21}} = 10^{-10} \tag{14}$$

$$U_2 \sim \frac{10^{21}}{10^{8}} = 10^{13} \tag{15}$$

$$U_3 \sim \sqrt{10^{-3}10^{-5}10^{8}10^{3}} \sim 10^{1} \tag{16}$$

$$U_4 \sim \frac{10^{-2}}{10^{8}} = 10^{-10} \tag{17}$$

Both U_1 and U_4 come close to the actual convective mantle velocity of 10^{-8}. Use of U_1 to nondimensionalise is probably best, since it has more degrees of freedom, and includes the gravity term that is probably important for such large masses.

3. **Problem Q2.**

3.1. *Statement*

Nondimensionalise N-S equation

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + \mu \boldsymbol{\nabla}^2 \mathbf{u} + \rho g \hat{\mathbf{z}} \tag{18}$$

where $\hat{\mathbf{z}}$ is the unit vector in the z direction. You may scale:

- velocity with the characteristic velocity U,
- time with R/U, where R is the characteristic length scale,
- pressure with ρU^2,

Reynolds number $\mathrm{Re} = RU\rho/\mu$ and Froude number $\mathrm{Fr} = gR/U$.

3.2. *Solution*

Let's start by dividing by $g\rho$, to make all terms (most obviously the $\hat{\mathbf{z}}$ term) dimensionless.

$$\frac{1}{g}\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{g}(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\frac{1}{g\rho}\boldsymbol{\nabla} p + \frac{\mu}{g\rho}\boldsymbol{\nabla}^2 \mathbf{u} + \hat{\mathbf{z}}. \tag{19}$$

Our suggested replacements are

$$\mathbf{u} = U\mathbf{u}' \tag{20}$$

$$\frac{\partial}{\partial t} = \frac{U}{R}\frac{\partial}{\partial t'} \tag{21}$$

$$p = \rho U^2 p' \tag{22}$$

$$\boldsymbol{\nabla} = \frac{1}{R}\boldsymbol{\nabla}'. \tag{23}$$

Plugging these in we have

$$\frac{U^2}{gR}\frac{\partial \mathbf{u}'}{\partial t'} + \frac{U^2}{gR}(\mathbf{u}' \cdot \boldsymbol{\nabla}')\mathbf{u}' = -\frac{\cancel{\rho} U^2}{g\cancel{\rho} R}\boldsymbol{\nabla}' p' + \frac{\mu U}{g\rho R^2}\boldsymbol{\nabla}'^2 \mathbf{u}' + \hat{\mathbf{z}}. \tag{24}$$

Making a $\mathrm{Fr} = gR/U$ replacement, using the Froude number, we have

$$\frac{U}{\mathrm{Fr}}\frac{\partial \mathbf{u}'}{\partial t'} + \frac{U}{\mathrm{Fr}}(\mathbf{u}' \cdot \boldsymbol{\nabla}')\mathbf{u}' = -\frac{U}{\mathrm{Fr}}\boldsymbol{\nabla}' p' + \frac{\mu}{\mathrm{Fr}\rho R}\boldsymbol{\nabla}'^2 \mathbf{u}' + \hat{\mathbf{z}}. \tag{25}$$

Scaling by Fr/U we tidy things up a bit, and also allow for insertion of the Reynold's number

$$\frac{\partial \mathbf{u}'}{\partial t'} + (\mathbf{u}' \cdot \boldsymbol{\nabla}')\mathbf{u}' = -\boldsymbol{\nabla}' p' + \frac{1}{\mathrm{Re}}\boldsymbol{\nabla}'^2 \mathbf{u}' + \frac{\mathrm{Fr}}{U}\hat{\mathbf{z}}. \tag{26}$$

Observe that the dimensions of Froude's number is that of velocity

$$[\mathrm{Fr}] = [g]T = \frac{L}{T}, \tag{27}$$

so that the end result is dimensionless as desired. We also see that Froude's number, characterizes the significance of the body force for fluid flow at the characteristic velocity. This is consistent with [1] where it was stated that the Froude number is used to determine the resistance of a partially submerged object moving through water, and permits the comparison of objects of different sizes (complete with pictures of canoes of various sizes that Froude built for such study).

4. Problem Q3.

4.1. *Statement*

In case of Stokes' boundary layer problem (see class note) calculate shear stress on the plate $y = 0$. What is the phase difference between the velocity of the plate $U(t) = U_0 \cos \omega t$ and the shear stress on the plate?

4.2. *Solution*

We found in class that the velocity of the fluid was given by

$$u(y,t) = U_0 e^{-\lambda y} \cos(\lambda y - \omega t) \tag{28}$$

where

$$\lambda = \sqrt{\frac{\omega}{2\nu}} \tag{29}$$

Calculating our shear stress we find

$$\mu \frac{\partial u}{\partial y} = U_0 \lambda \mu e^{-\lambda y} \left(-1 - \sin(\lambda y - \omega t)\right)$$

and on the plate ($y = 0$) this is just

$$\mu \frac{\partial u}{\partial y}\bigg|_{y=0} = U_0 \lambda \mu (-1 + \sin(\omega t)). \tag{30}$$

We've got a constant term, plus one that is sinusoidal. Observing that

$$\cos x = \mathrm{Re}(e^{ix}) \tag{31}$$

$$\sin x = \mathrm{Re}(-ie^{ix}) = \mathrm{Re}(e^{i(x-\pi/2)}), \tag{32}$$

The phase difference between the non-constant portion of the shear stress at the plate, and the plate velocity $U(t) = U_0 \cos \omega t$ is just $-\pi/2$. The shear stress at the plate lags the driving velocity by 90 degrees.

References

[1] Wikipedia. Froude number — wikipedia, the free encyclopedia [online]. 2012. [Online; accessed 27-March-2012]. Available from: http://en.wikipedia.org/w/index.php?title=Froude_number&oldid=479498080. 3.2