Derivative recurrence relation for Hermite polynomials.

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1. Motivation.

For a QM problem I had need of a recurrence relation for Hermite polynomials. I found it in [1], but thought I'd try to derive the relation myself.

2. Guts

The starting point I'll use is the Rodgigues' formula definition of the Hermite polynomials

$$H_n = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \tag{1}$$

Let's write $D = d/dx$, and take the derivative of H_n

$$\begin{aligned}
DH_n &= D\left(e^{x^2} D^n e^{-x^2}\right) \\
&= 2xe^{x^2} D^n e^{-x^2} + e^{x^2} D^n \left(-2xe^{-x^2}\right) \\
&= 2xe^{x^2} D^n e^{-x^2} + e^{x^2} \sum_{k=0}^{n} \binom{n}{k} D^k(-2x) D^{n-k} e^{-x^2} \\
&= 2xe^{x^2} D^n e^{-x^2} + e^{x^2} \sum_{k=0}^{1} \binom{n}{k} D^k(-2x) D^{n-k} e^{-x^2} \\
&= \cancel{2xe^{x^2} D^n e^{-x^2}} + e^{x^2} \left(\cancel{-2xD^n e^{-x^2}} - 2nD^{n-1} e^{-x^2}\right)
\end{aligned}$$

So we have the rather simple end result

$$\frac{d}{dx} H_n(x) = 2nH_{n-1}(x) \tag{2}$$

References

[1] M. Abramowitz and I.A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55. Dover publications, 1964. 1